Hybrid discontinuous Galerkin discretizations of Galbrun's equation

Martin Halla[1], Christoph Lehrenfeld[2], Tim van Beeck[2]

[1] Johann Radon Institute for Computational and Applied Mathematics, Linz
[2]Institute for Numerical and Applied Mathematics, University of Göttingen

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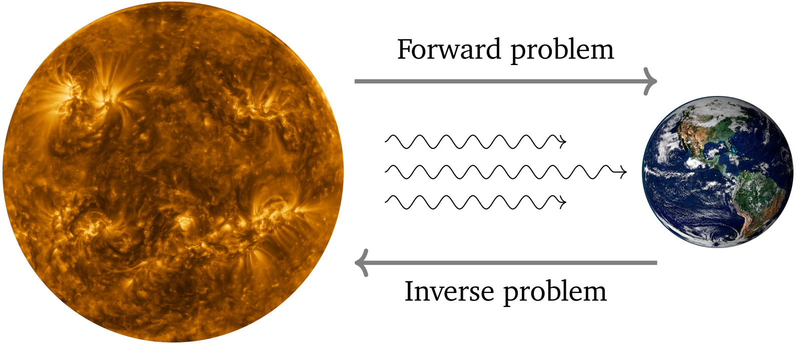
Motivation: Helioseismology

- Helioseismology $\widehat{=}$ study of the Sun through its *oscillations*



- Forward problem → Galbrun's equation describes time-harmonic waves in the presence of a steady background flow

Galbrun's equation

Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, s.t.

$$-\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2\boldsymbol{u} - \nabla\left(c_s^2\rho\,\mathrm{div}\,\boldsymbol{u}\right) + (\mathrm{div}\,\boldsymbol{u})\nabla p - \nabla(\nabla p \cdot \boldsymbol{u}) \\ + (\mathrm{Hess}(p) - \rho\,\mathrm{Hess}(\phi))\boldsymbol{u} - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O},$$

ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient

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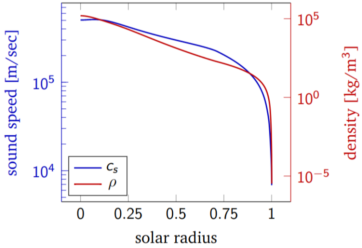
Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, (assuming that $p = \text{const.}$, $\phi = \text{const.}$) s.t.

$$-\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2\boldsymbol{u} - \nabla\left(c_s^2\rho\operatorname{div}\boldsymbol{u}\right) - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O},$$

ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient

Challenges

- nonstandard differential operator $\partial_{\boldsymbol{b}} := \sum_{l=1}^{d} \boldsymbol{b}_l \partial_{x_l}$
- indefinite problem
- highly varying coefficients
- computational expensive (vector valued, ...)



Abstract tools: (weak) T-coercivity

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Let X be Hilbert, $A \in L(X)$ and $f \in X'$. Then, the problem of finding $u \in X$ s.t. $Au = f$ is well-posed if and only if A^* is injective and the inf-sup condition holds

$$\inf_{u \in X} \sup_{v \in X} \frac{|\langle Au, v \rangle_X|}{\|u\|_X \|v\|_X} \geq \beta > 0$$

T-coercivity (Definition & Thm.)[1]

We call an operator $A \in L(X)$ *T-coercive* if there exists a bijective operator $T \in L(X)$ s.t. AT is coercive, i.e. $\Re\{\langle ATu, u \rangle_X\} \geq \alpha \|u\|_X^2$ for all $u \in X$.
The problem $Au = f$ is well-posed if and only if A is T-coercive.

$A \in L(X)$ is called *weakly T-coercive* if there $\exists$ $T \in L(X)$ bijective and $K \in L(X)$ compact s.t. $AT + K$ is coercive.

→ weak T-coercivity $\Rightarrow$ Fredholm with index 0 $\Rightarrow$ well-posedness $\Leftrightarrow$ injectivity

[1] see e.g. P.Ciarlet Jr., *T-coercivity: Application to the discretization of Helmholtz-like problems.* CAMWA, 2012.

Abstract tools: (Weak) T-compatibility

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For conforming FEM where $X_n \subset X$, $A_n = p_n A|_{\mathbb{X}_n}$ ($p_n : X \to X_n$ orthogonal projection), stability is inherited to the discrete level if there exists a sequence of bijective operators $T_n \in L(X_n)$, $n \in \mathbb{N}$, s.t.

$$\lim_{n\to\infty} \|T - T_n\|_{L(X)} = 0$$

➔ non-conforming case? weak T-coercivity?

Setting

X Hilbert space, $(X_n)_{n\in\mathbb{N}}$ sequence of Hilbert spaces s.t. possibly $X_n \not\subset X$.
We assume the existences of a family $(p_n)_{n\in\mathbb{N}}$, $p_n : X \to X_n$ s.t.

$$\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X.$$

We write $u_n \xrightarrow{P} u$ if $\lim_{n\to\infty} \|p_n u - u_n\|_{X_n} = 0$ and for $A_n \in L(X_n)$, $A \in L(X)$, we write $A_n \xrightarrow{P} A$ if $\lim_{n\to\infty} \|A_n p_n u - p_n A u\|_{X_n}$ for all $u \in X$.

Abstract tools: (Weak) T-compatibility

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Theorem (Weak T-compatibility[2])

Let $A \in L(X)$ be weakly T-coercive & injective; $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, be s.t. $A_n \xrightarrow{P} A$. If $\exists$ uniformly bounded sequences $(T_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ s.t. T_n and B_n are uniformly stable, $(K_n)_{n\in\mathbb{N}}$ compact, $A_nT_n = B_n + K_n$ and

$$\lim_{n\to\infty} \|T_n p_n u - p_n Tu\|_{X_n} = 0, \qquad \lim_{n\to\infty} \|B_n p_n u - p_n Bu\|_{X_n} = 0 \qquad \forall u \in X$$

Then $(A_n)_{n\in\mathbb{N}}$ is uniformly stable, i.e. A_n^{-1} exists and $\|A_n^{-1}\|_{L(X_n)} \leq C$ for all $n > n_0$.

$$\begin{array}{ccccccc} X & A & T & = & B & + & K \\ \downarrow p_n & \xrightarrow{P}\uparrow & \xrightarrow{P}\uparrow & & \xrightarrow{P}\uparrow & & \\ X_n & A_n & T_n & = & B_n & + & K_n \end{array}$$

→ transfer (weakly T-coercive) structure to the discrete level in a stable manner

[2] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility condition and its application to the [discr.] of … Galbrun's equation*. arXiv, 2022.

Continuous weak formulation

We define

$$\mathbb{X} := \{\boldsymbol{u} \in \boldsymbol{L}^2 : \operatorname{div} \boldsymbol{u} \in L^2, \partial_{\boldsymbol{b}} \boldsymbol{u} \in \boldsymbol{L}^2, \boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\mathcal{O}\},$$

$$\langle \boldsymbol{u}, \boldsymbol{u}' \rangle_{\mathbb{X}} := \langle \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle + \langle \boldsymbol{u}, \boldsymbol{u}' \rangle + \langle \partial_{\boldsymbol{b}} \boldsymbol{u}, \partial_{\boldsymbol{b}} \boldsymbol{u}' \rangle$$

Weak formulation

Find $\boldsymbol{u} \in \mathbb{X}$ s.t. $\langle A\boldsymbol{u}, \boldsymbol{u}' \rangle_{\mathbb{X}} := a(\boldsymbol{u}, \boldsymbol{u}') = \langle \boldsymbol{f}, \boldsymbol{u}' \rangle$ for all $\boldsymbol{u}' \in \mathbb{X}$, where

$$a(\boldsymbol{u}, \boldsymbol{u}') = \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega \times)\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega \times)\boldsymbol{u}' \rangle - i\omega \langle \rho\gamma \boldsymbol{u}, \boldsymbol{u}' \rangle$$

Indefiniteness: if $\boldsymbol{u} \in \ker(\operatorname{div})$ then

$$\Re\{a(\boldsymbol{u}, \boldsymbol{u})\} = - \|\rho^{1/2}(\omega + i\partial_{\boldsymbol{b}} + i\Omega \times)\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 \not\gtrsim \|\boldsymbol{u}\|_{\mathbb{X}}^2.$$

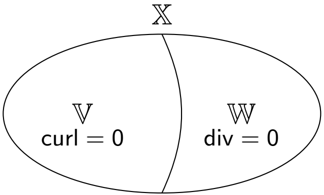
➔ $a(\cdot, \cdot)$ is not coercive!

Construction of T

Goal: Construct $T \in L(\mathbb{X})$ s.t. $a(T\cdot,\cdot)$ (+compact) is coercive.
Idea: Flip the problematic sign! For $\boldsymbol{u} \in \mathbb{X}$, find $v \in H^2_*$ .s.t.

$$\operatorname{div} \nabla v = \operatorname{div} \boldsymbol{u} \text{ in } \mathcal{O},$$
$$\boldsymbol{\nu} \cdot \nabla v = 0 \text{ on } \partial\mathcal{O}$$

and set $\boldsymbol{v} = \nabla v$, $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, $T\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{w}$.



→ If $\boldsymbol{u} \in \ker(\operatorname{div})$, i.e. $\boldsymbol{u} = \boldsymbol{w}$, then

$$\Re\{a(T\boldsymbol{u},\boldsymbol{u})\} = \|\rho^{1/2}(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}\|^2_{\boldsymbol{L}^2} \gtrsim \|\boldsymbol{u}\|^2_{\mathbb{X}}$$

→ For a stable discretization, we want to transfer this decomposition to the discrete level in a stable manner!

Interlude: Stability of the Stokes complex

Excerpt from the De Rham complex (2D):

$$\begin{array}{ccc} \boldsymbol{H}^1 & \xrightarrow{div} & L^2. \\ \downarrow & & \downarrow \\ \mathbb{X}_n & \xrightarrow{div} & Q_n \end{array}$$

Stability of div only for specific choices of $\mathbb{X}_n$, Q_n

- Scott-Vogelius: $\mathbb{X}_n = [\mathbb{P}^k(\mathcal{T}_n)]^d \cap \boldsymbol{H}^1$, $Q_n = \mathbb{P}^k(\mathcal{T}_n)$
  - stable for $k \geq k_0$ with $k_0 = 4$ (2D), $k_0 = 8$ (3D)
  - with special meshes (barycentric refs): $k_0 = 2$ (2D)
- (H(div)-conforming-) DG discretizations less restrictive in k

Discretizations of Galbrun's equation

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→ Stability of Galbrun's equation is connected to the stability of the Stokes problem, e.g. an $\boldsymbol{H}^1$-conforming discretization[3]

$$\mathbb{X}_n := \{\boldsymbol{u}_n \in \boldsymbol{L}^2 : \boldsymbol{u}_n|_\tau \in \mathcal{P}^k(\tau) \quad \forall \tau \in \mathcal{T}_n, \boldsymbol{\nu} \cdot \boldsymbol{u}_n = 0 \text{ on } \partial\mathcal{O}\} \cap \boldsymbol{H}^1(\mathcal{O})$$

- requires $k \geq 4$ in 2D or barycentric refinement, $k \geq 6$ for uniform tetrahedral meshes (3D),
- Mach number must be bounded suitably:

$$\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \lesssim \beta_h^2 \frac{\min\{c_s^2\rho\}}{\max\{c_s^2\rho\}}$$

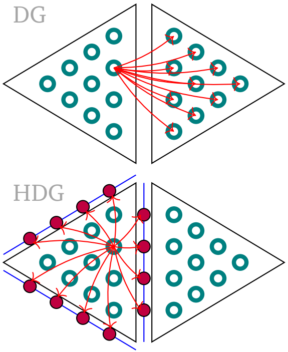
→ Stability and Mach number requirements improved for $H(\mathrm{div})$-conforming DG[4]

[3] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility condition and its application to the [discr.] of ... Galbrun's equation*. arXiv, 2022.

[4] M. Halla, *Convergence analysis of nonconform H(div)-finite elements for the damped time-harmonic Galbrun's equation* arXiv, 2023.

Hybrid DG[5,6]

Idea: introduce additional facet variable to enforce a "good" sparsity pattern & leverage static condensation



Schur complement with $S = A_{\mathcal{T}_n\mathcal{T}_n} - A_{\mathcal{T}_n\mathcal{F}_n} A^{-1}_{\mathcal{F}_n\mathcal{F}_n} A_{\mathcal{F}_n\mathcal{T}_n}$

$$\begin{pmatrix} A_{\mathcal{T}_n\mathcal{T}_n} & A_{\mathcal{T}_n\mathcal{F}_n} \\ A_{\mathcal{F}_n\mathcal{T}_n} & A_{\mathcal{F}_n\mathcal{F}_n} \end{pmatrix} = \begin{pmatrix} I & A_{\mathcal{T}_n\mathcal{F}_n} A^{-1}_{\mathcal{F}_n\mathcal{F}_n} \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} S & 0 \\ 0 & A_{\mathcal{F}_n\mathcal{F}_n} \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ A^{-1}_{\mathcal{F}_n\mathcal{F}_n} A_{\mathcal{T}_n\mathcal{F}_n} & I \end{pmatrix}$$

[5] B. Cockburn, J. Gopalakrishnan, R. Lazarov, *Unified hybrdization of [dG] … for [2nd] order elliptic problems*. SINUM, 2009

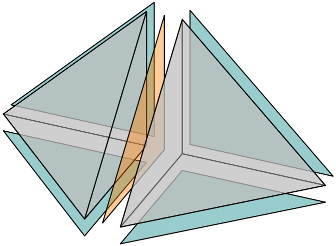
[6] C. Lehrenfeld, *[HDG] Methods for Incompressible Flow Problems*. Diploma thesis, 2010.

Hybrid DG discretization

Let $\mathcal{T}_n$ be a triangulation of $\mathcal{O}$ and let $\mathcal{F}_n$ be the set of all faces of $\mathcal{T}_n$. Set $\mathbb{X}_n := \mathbb{X}_{\mathcal{T}_n} \times \mathbb{X}_{\mathcal{F}_n}$ where

- $\mathbb{X}_{\mathcal{T}_n} = \mathbb{P}^k(\mathcal{T}_n), \mathbb{BDM}^k(\mathcal{T}_n), \ldots$
- $\mathbb{X}_{\mathcal{F}_n} = \mathbb{P}^k(\mathcal{F}_n), \mathbb{P}^{k,\mathrm{tang}}(\mathcal{F}_n), \ldots$

Notation: $\langle \cdot, \cdot \rangle_{\mathcal{T}_n} := \sum_{\tau \in \mathcal{T}_n} \langle \cdot, \cdot \rangle_{L^2(\tau)}$, $\langle \cdot, \cdot \rangle_{\mathcal{F}_n} := \ldots$



For a tuple $\boldsymbol{u}_n = (\boldsymbol{u}_\tau, \boldsymbol{u}_F) \in \mathbb{X}_n$, we define

$$[\![\underline{\boldsymbol{u}_n}]\!] := \boldsymbol{u}_\tau - \boldsymbol{u}_F, \quad [\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{\nu}} = \boldsymbol{\nu} \cdot [\![\underline{\boldsymbol{u}_n}]\!], \quad [\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{b}} = (\boldsymbol{b} \cdot \boldsymbol{\nu})[\![\underline{\boldsymbol{u}_n}]\!]$$

Stabilization & Lifting operators

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- symmetric interior penalty: penalty param. α has to be chosen large enough
- → problematic for convection term, leads to (further) restrictions in $\|c_s^{-1}\boldsymbol{b}\|_{L^\infty}^2$

Lifting operators (Bassi-Rebay stabilization[7])

For $\boldsymbol{u}_n \in \mathbb{X}_n$, define $\boldsymbol{R}^l\boldsymbol{u}_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ and $R^l\boldsymbol{u}_n \in \mathbb{P}^l(\mathcal{T}_n)$ as

$$\langle \boldsymbol{R}^l\boldsymbol{u}_n, \psi_n\rangle_{\mathcal{T}_n} = -\langle [\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{b}}, \psi_n\rangle_{\partial\mathcal{T}_n} \qquad \forall\psi_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d,$$
$$\langle R^l\boldsymbol{u}_n, \psi_n\rangle_{\mathcal{T}_n} = -\langle [\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{\nu}}, \psi_n\rangle_{\partial\mathcal{T}_n} \qquad \forall\psi_n \in \mathbb{P}^l(\mathcal{T}_n).$$

Define the discrete differential operators elementwise for $\tau \in \mathcal{T}_n$

$$(\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{u}_n)\,|_\tau := \partial_{\boldsymbol{b}}(\boldsymbol{u}_n)|_\tau + \boldsymbol{R}^l\boldsymbol{u}_n|_\tau$$
$$(\mathrm{div}_{\boldsymbol{\nu}}^n\,\boldsymbol{u}_n)\,|_\tau := \mathrm{div}(\boldsymbol{u}_n)|_\tau + R^l\boldsymbol{u}_n|_\tau$$

[7] F. Bassi, S. Rebay, *A high-order accurate [disc. FEM] for the numerical [sol.] of the compressible Navier–Stokes [eqs.]*. Journal of Comp. Physics, 1997.

Discrete weak formulation

For $\boldsymbol{u}_n, \boldsymbol{u}_n' \in \mathbb{X}_n$, we define the scalar product (and associated norm $\|\cdot\|^2_{\mathbb{X}_n} := \langle\cdot,\cdot\rangle_{\mathbb{X}_n}$)

$$\langle \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\mathbb{X}_n} := \langle \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} + \langle \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} + \langle \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}_n, \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} + \langle h_\tau^{-1} [\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{\nu}}, [\![\underline{\boldsymbol{u}_n'}]\!]_{\boldsymbol{\nu}} \rangle_{\partial\mathcal{T}_n}.$$

Discrete weak formulation

Find $\boldsymbol{u}_n \in \mathbb{X}_n$ s.t. $\langle A_n \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\mathbb{X}_n} := a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}, \boldsymbol{u}_n' \rangle$ for all $\boldsymbol{u}_n' \in \mathbb{X}_n$, where

$$\begin{aligned} a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') :=& \langle c_s^2 \rho \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} - \langle \rho(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} \\ & - i\omega \langle \gamma\rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} + s_n(\boldsymbol{u}_n, \boldsymbol{u}_n') \end{aligned}$$

and

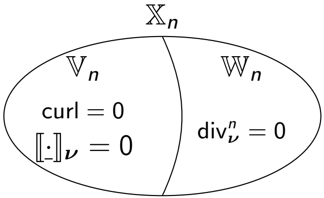
$$s_n(\boldsymbol{u}_n, \boldsymbol{u}_n') := \langle c_s^2 \rho \frac{\alpha}{h_\tau} [\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{\nu}}, [\![\underline{\boldsymbol{u}_n'}]\!]_{\boldsymbol{\nu}} \rangle_{\partial\mathcal{T}_n} - \langle c_s^2 \rho R^l \boldsymbol{u}_n, R^l \boldsymbol{u}_n \rangle_{\mathcal{T}_n}$$

- ➔ BR for convection term, SIP for diffusion term!
- ➔ $\operatorname{div}^n_{\boldsymbol{\nu}}$ only introduced for (convenient) notation

Construction of T_n

Goal: construct decomposition $\mathbb{X}_n \ni \boldsymbol{u}_n = \boldsymbol{v}_n + \boldsymbol{w}_n$ and set $T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n$ s.t.

$$\operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{v}_n = \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \quad [\![\underline{\boldsymbol{v}_n}]\!]_{\boldsymbol{\nu}} = 0, \quad \|\boldsymbol{v}_n\|_{\mathbb{X}_n} \leq C\|\boldsymbol{u}_n\|_{\mathbb{X}_n}$$



Helmholtz decomposition on the discrete level

For $\boldsymbol{u}_n \in \mathbb{X}_n$, find $\tilde{v} \in H^2_*$ s.t.

$$\operatorname{div} \nabla \tilde{v} = \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n \text{ in } \mathcal{O},$$
$$\boldsymbol{\nu} \cdot \nabla \tilde{v} = 0 \text{ on } \partial\mathcal{O}$$

→ next step: project $\nabla \tilde{v}$ into $\mathbb{X}_n$

Construction of T_n

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For $s > 1/2$, let $\pi_n^d : \boldsymbol{H}^s \to [\mathbb{P}^k(\mathcal{T}_n)]^d \cap H(\mathrm{div})$ be an $H(\mathrm{div})$-conforming interpolation operator. We extend the operator to the HDG setting:

$$\underline{\pi_n^d}\boldsymbol{u} := (\pi_n^d \boldsymbol{u}, P_{\boldsymbol{\nu}} \pi_n^d \boldsymbol{u} + P_{\boldsymbol{\nu}}^{\perp} \pi_n^F \boldsymbol{u}),$$

where

- $P_{\boldsymbol{\nu}}$ / $P_{\boldsymbol{\nu}}^{\perp}$ is the normal / tangential projection on a facet $F \in \mathcal{F}_n$,
- π_n^F is the L^2-orthogonal projection on the facet.

Definition of T_n

For $\boldsymbol{u}_n \in \mathbb{X}_n$, we set $\boldsymbol{v}_n := \underline{\pi_n^d}\nabla\tilde{v}$, $\boldsymbol{w}_n := \boldsymbol{u}_n - \boldsymbol{v}_n$ and $T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n$. It holds

- $[\![\underline{\boldsymbol{v}_n}]\!]_{\boldsymbol{\nu}} = \pi_n^d \nabla \tilde{v} \cdot \boldsymbol{\nu} - \pi_n^d \nabla \tilde{v} \cdot \boldsymbol{\nu} = 0$ ✓
- $\mathrm{div}_{\boldsymbol{\nu}}^n \boldsymbol{v}_n = \mathrm{div}\, \pi_n^d \nabla \tilde{v} = \pi_n^l \mathrm{div}\, \tilde{v} = \mathrm{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n$ ✓
- $\|\boldsymbol{v}_n\|_{\mathbb{X}_n} \leq C \|\boldsymbol{u}_n\|_{\mathbb{X}_n}$ ✓

Convergence results

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Can show that $\exists\ (p_n)_{n\in\mathbb{N}}$, $p_n \in L(\mathbb{X}_n)$, s.t. $\lim_{n\to\infty} \|p_n u\|_{\mathbb{X}_n} = \|u\|_{\mathbb{X}}$ & $A_n \xrightarrow{P} A$.
Furthermore, $(T_n)_{n\in\mathbb{N}}$ is stable and $T_n \xrightarrow{P} T$.

Theorem

Assume that $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \lesssim 1$ *and* $\alpha > 0$ *is large enough. Then,* $\exists\ (B_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ *s.t.* $(B_n)_{n\in\mathbb{N}}$ *is uniformly coercive,* $(K_n)_{n\in\mathbb{N}}$ *is compact,* $B_n \xrightarrow{P} B$ *and* $A_n T_n = B_n + K_n$.

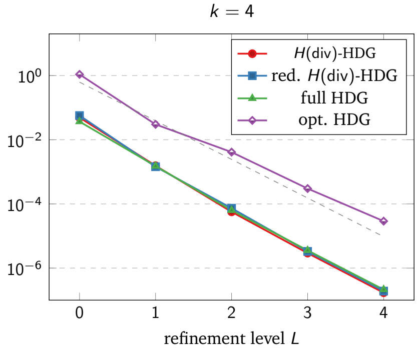
Corollary

Under the assumptions from above, $\exists n_0 > 0$ *s.t. the discrete problem has a unique solution for all* $n > n_0$. *If* $\boldsymbol{u} \in \mathbb{X} \cap \boldsymbol{H}^{s+2}$, $s > 0$, $\rho \in W^{1,\infty}$, *and* $\boldsymbol{b} \in \boldsymbol{W}^{1,\infty}$, *then*

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \leq C \left(h^{\min\{1+s,k\}} + h^{\min\{k,l+1\}} \right) \|\boldsymbol{u}\|_{\boldsymbol{H}^{s+2}}.$$

- $\mathbb{X}_{\mathcal{T}_n}$ and $\mathbb{X}_{\mathcal{F}_n}$ not specified before
- different options to optimize the computational efficiency

HDG method	discrete spaces			associated costs		
	$\mathbb{X}_{\mathcal{T}_n}$	$\mathbb{X}_{\mathcal{F}_n}$	lifting	ndofs	ncdofs	nze
full	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	$[\mathbb{P}^k(\mathcal{F}_n)]^d$	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	124	20	784
$H(\mathrm{div})$[8]	$\mathbb{BDM}^k(\mathcal{T}_n)$	$[\mathbb{P}^{k,\mathrm{tang}}(\mathcal{F}_n)]^d$	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	88	20	784
red. $H(\mathrm{div})$[9]	$\mathbb{BDM}^k(\mathcal{T}_n)$	$[\mathbb{P}^{k-1,\mathrm{tang}}(\mathcal{F}_n)]^d$	$[\mathbb{P}^{k-1}(\mathcal{T}_n)]^d$	51	15	441
optimized[10]	$\mathbb{BDM}_k^-(\mathcal{T}_n)$	$[\mathbb{P}^{k-1,\mathrm{tang}}(\mathcal{F}_n)]^d$	$[\mathbb{P}^{k-1}(\mathcal{T}_n)]^d$	56	10	308



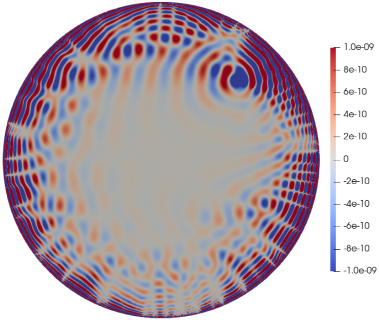
[8] C. Lehrenfeld, J. Schöberl. *High order exactly [div.-free HDG] methods for unsteady [incompr.] flows.* Comput. Methods Appl. Mech. Eng., 2016.

[9] similar to the *projected jumps* modification[6]

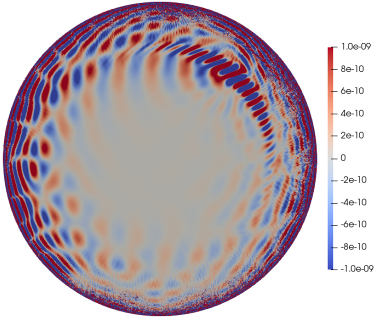
[10] P. L. Lederer, C. Lehrenfeld, J. Schöberl. *[HDG] methods with relaxed H(div)-conformity for incompressible flows. Part I.* SIAM J. Numer. Anal., 2018.

Numerical examples: Sun coefficients

➔ c_s, ρ, p taken from `modelS`[11], $\omega = 0.003 \cdot 2\pi \cdot R_{\text{sun}}$, $\gamma = \omega/100$, $\Omega = (0,0)^T$,
rhs $\boldsymbol{f} = 10^7 \cdot (g, 0)^T$ & flow $\boldsymbol{b} = c_s / R_{\text{sun}} \cdot c_{\boldsymbol{b}} \cdot (-y, x)^T$.



$\|c_s^{-1}\boldsymbol{b}\|_{L^\infty}^2 \approx 0.05$



$\|c_s^{-1}\boldsymbol{b}\|_{L^\infty}^2 \approx 1.00$

[11] J. Christensen-Dalsgaard et al., *The current state of solar modeling*. Science, 1996.

Conclusions

- Stability of Galbrun's equation related to the stability of the Stokes problem;
- (H)DG discretization is stable for $k \geq 1$, bound on Mach number less restrictive than $\boldsymbol{H}^1$-conforming discretization
  - → Analysis through (weak) T-coercivity / T-compatibility arguments;
  - → can be extended to the full equation by adjusting the Helmholtz decomposition

$$(\mathrm{div} + P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v} = (\mathrm{div}^n_{\boldsymbol{\nu}} + \pi^l_n\boldsymbol{q}\cdot + M_n)\boldsymbol{u}_n.$$

- Hybridization reduces computational effort;
- Method is robust w.r.t. to (drastic) changes in the coefficients!

Thank you for your attention!