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GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

Unfitted Mixed Finite Element Methods

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Background: Unfitted FEM

Problems

PDEs on embedded surfaces, moving domains, separate geometry description

Challenges

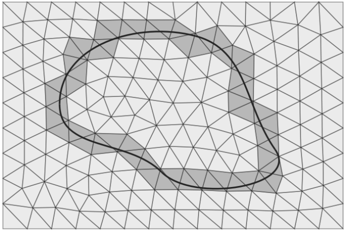
Stability / Robustness w.r.t. arbitrarily small cuts

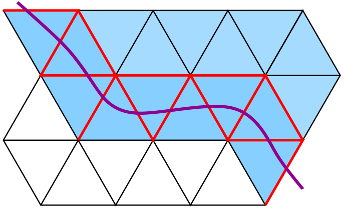
↓

Ghost penalty stabilization[1]

$$\text{👻}(u_h, v_h) := \sum_{F \in \mathcal{F}_h^{\partial\Omega}} \sum_{l=0}^{k} \frac{h_F^{2l+1}}{l!^2} \int_F [\![\partial_n^l u_h]\!][\![\partial_n^l v_h]\!] ds$$

Requirement: $\|u\|_{H^q(\Omega)} + |u|_{\text{👻}} \simeq \|u\|_{H^q(\Omega^{\mathcal{T}})}$





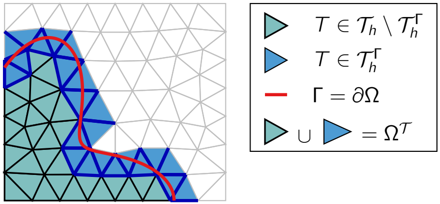
[1] E. Burman. Ghost penalty. C.R. Math., 348(21-22):1217–1220, November 2010.

Mixed Poisson problem

Model problem (Dirichlet case)

Find u, p with $p = p_D$ on $\partial\Omega$ s.t.

$$\begin{aligned} u - \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div} u &= -f \quad \text{in } \Omega. \end{aligned}$$



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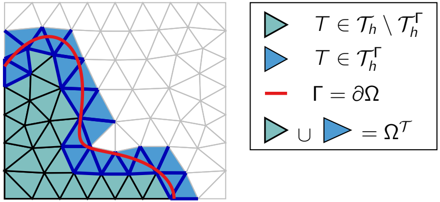

Weak formulation - fitted mixed Poisson

Find $u_h \in \Sigma_h = \mathbb{RT}^k \subset H(\operatorname{div}, \Omega)$, $p_h \in Q_h = \mathbb{P}^k \subset L^2(\Omega)$ s.t.

$$\begin{aligned} (u_h, v_h)_\Omega + (\operatorname{div} v_h, p_h)_\Omega &= (v_h, p_D)_{\partial\Omega} \qquad \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_\Omega \qquad\qquad &= (-f, q_h)_\Omega \qquad \forall q_h \in Q_h. \end{aligned}$$

Mixed Poisson problem

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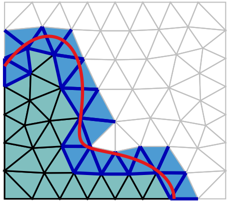

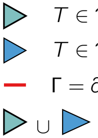

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Weak formulation - naive unfitted mixed Poisson

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h) \subset H(\operatorname{div}, \Omega^\mathcal{T})$, $p_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h) \subset L^2(\Omega^\mathcal{T})$ s.t.

$$(u_h, v_h)_\Omega + (\operatorname{div} v_h, p_h)_\Omega = (v_h, p_D)_{\partial\Omega} \qquad \forall v_h \in \Sigma_h,$$
$$(\operatorname{div} u_h, q_h)_\Omega = (-f, q_h)_\Omega \qquad \forall q_h \in Q_h.$$

➔ Not robust! inf-sup stability depends on cut position

Mixed Poisson problem

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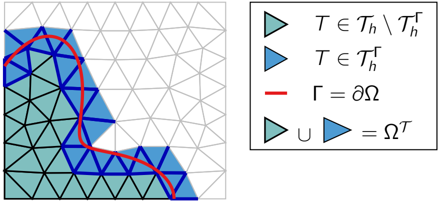

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$$\begin{aligned} (u_h, v_h)_\Omega + (\operatorname{div} v_h, p_h)_\Omega + \text{👻} &= (v_h, p_D)_{\partial\Omega} \qquad \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_\Omega + \text{👻} \qquad\qquad\qquad &= (-f, q_h)_\Omega \qquad \forall q_h \in Q_h. \end{aligned}$$

- ➔ Not robust! inf-sup stability depends on cut position
- ➔ Ghost penalty pollutes mass balance!

Robust Unfitted Mixed Poisson

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Observation

$$\{u_h \in \Sigma_h \mid (\operatorname{div} u_h, q_h)_{\Omega} = 0 \ \forall q_h \in Q_h\} = \{u_h \in \Sigma_h \mid (\operatorname{div} u_h, q_h)_{\Omega^{\mathcal{T}}} = 0 \ \forall q_h \in Q_h\}$$

➔ Kernel-coercivity of $(u_h, v_h)_{\Omega}$-block unaffected

Modification

For $\gamma_{\text{👻}} \geq 0$, find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$ s.t.

$$\begin{aligned} (u_h, v_h)_{\Omega} + \gamma_{\text{👻}} \text{👻}(u_h, v_h) + (\operatorname{div} v_h, \bar{p}_h)_{\Omega^{\mathcal{T}}} &= (v_h, p_D)_{\partial\Omega} \qquad && \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_{\Omega^{\mathcal{T}}} &= (-f_h, q_h)_{\Omega^{\mathcal{T}}} && \forall q_h \in Q_h. \end{aligned}$$

➔ Inf-sup stable independent of cut position, consistent for u (up to $f_h \approx f$), consistent for p on interior elements, but not on cut elements

➔ Error estimate: $\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} + \|u^{\mathcal{E}} - u_h\|_{\Sigma} + |u_h|_{\text{👻}} \lesssim h^m \|u\|_{H^m(\Omega)} + \mathtt{err}_f$

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Motivation

Apply post-processing to

- repair inconsistency on cut elements
- obtain higher order convergence

Idea

Make use of the relation $\nabla p = u$ and exploit the accuracy of u_h.

Two versions

→ element-wise post-processing

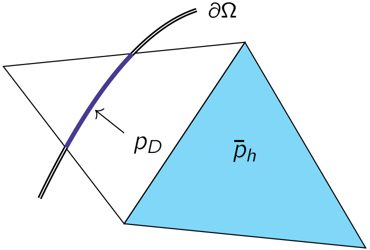
→ patchwise post-processing

Elementwise Post Processing

Element-local Scheme

For each $T \in \mathcal{T}_h$, find $p_h^* \in \mathbb{P}^{k+1}(T)$ s.t.

$$\begin{aligned}
(\nabla p_h^*, \nabla q_h^*)_T &= (u_h, \nabla q_h^*)_T \quad \forall q_h^* \in \mathbb{P}^{k+1}(T) \setminus \mathbb{R}, \\
(p_h^*, 1)_T &= (\bar{p}_h, 1)_T \quad \text{if } T \in \mathcal{T}_h^{\text{interior}}, \\
(p_h^*, 1)_{T\cap\partial\Omega} &= (p_D, 1)_{T\cap\partial\Omega} \quad \text{if } T \in \mathcal{T}_h^{\text{cut}}.
\end{aligned}$$

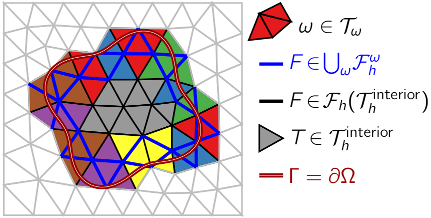


Error estimate

If $\gamma_{👻} > 0$ and $p^{\mathcal{E}} \in H^{k+2}(\Omega^{\mathcal{T}})$, then

$$\|p^{\mathcal{E}} - p_h^*\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h^{k+2} \|p^{\mathcal{E}}\|_{H^{k+2}(\Omega^{\mathcal{T}})}.$$

Patchwise Post Processing



Patchwise Scheme

For each $\omega \in \mathcal{T}_\omega$, find $p_h^* \in \mathbb{P}^{k+1}(\omega)$ s.t. for all $q_h^* \in \mathbb{P}^{k+1}(\omega) \setminus \mathbb{R}$:

$$(\nabla p_h^*, \nabla q_h^*)_{\Omega\cap\omega} + \text{👻}_\omega(p_h^*, q_h^*) = (u_h, \nabla q_h^*)_{\Omega\cap\omega},$$
$$(p_h^*, 1)_{\Omega^{\text{interior}}\cap\omega} = (\bar{p}_h, 1)_{\Omega^{\text{interior}}\cap\omega}.$$

Error estimate

For $p \in H^{k+2}(\Omega)$, it holds

$$\|p - p_h^*\|_{L^2(\Omega)} \lesssim h^{k+2}\|p\|_{H^{k+2}(\Omega)}$$

- ➔ $\gamma_{\text{👻}} = 0$ allowed (hybridization possible)
- ➔ no dependence on Dirichlet boundary data

Numerical Example

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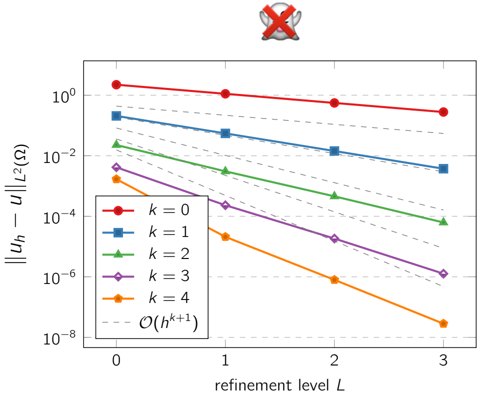
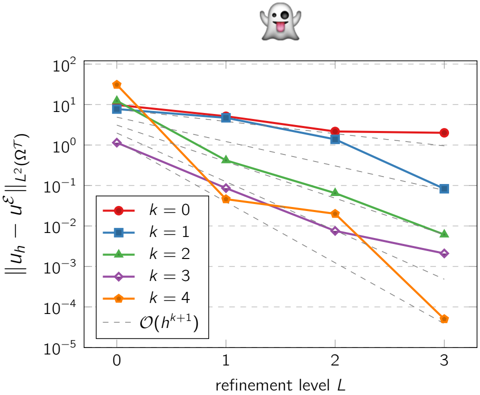
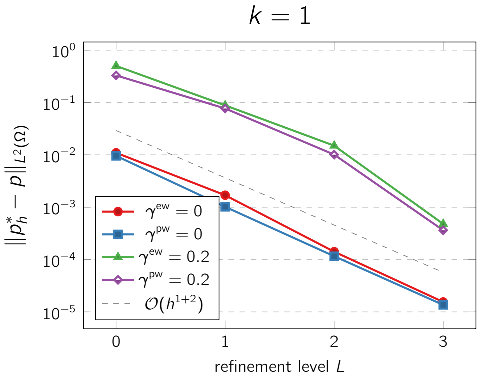
Convergence of u_h

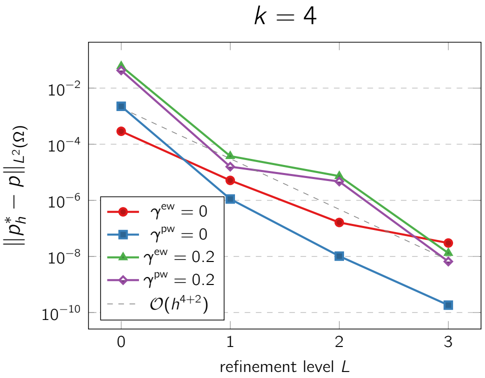




Numerical Example

Elementwise vs. Patchwise Postprocessing





Conclusion & Outlook

Unfitted mixed FEM

- Inf-sup stability without polluting the mass balance
- Recover higher order convergence of p_h with post processing
  - → Element-wise post processing: requires Dirichlet boundary data, $\gamma_{\text{👻}} > 0$ necessary
  - → Patchwise post processing: no dependence on boundary data, $\gamma_{\text{👻}} = 0$ allowed
- Hybridization possible
  - → Caution: conditioning

Extensions

- Neumann boundary conditons
- Hybridization
- Stokes

Backup: Literature

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Poisson / Darcy Problem

Mass balance polluted, stability:

R. Puppi, *A cut finite element method for the Darcy problem.* arXiv: 2111.09922, 2021.

P. Cao, J. Chen, *An extended finite element method for coupled Darcy-Stokes problems.* IJNME, 2022

Mass balance hardly polluted, stability, divergence preserving:

T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A divergence preserving cut finite element method for Darcy flow.* arXiv: 2205.12023, 2022.

Stokes Problem

T. Frachon, E. Nilsson, Sara Zahedi, *Divergence-free cut finite element methods for Stokes flow.* arXiv:2304.14230, 2023.

Similar Method:

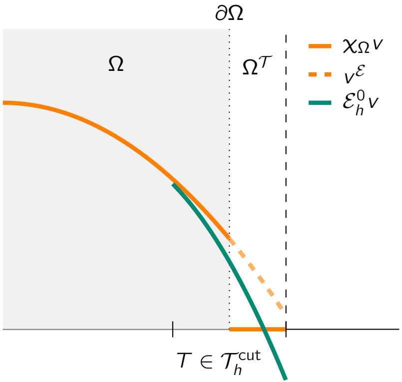
E. Burman, P. Hansbo, M. G. Larson, *Cut finite element method for divergence free approximation of incompressible flow: optimal error estimates and pressure independence.* arXiv: 2207.04734, 2022.

Backup: Interpretation of $\bar{p}_h$

Define $\mathcal{E}_h^0 : L^2(\Omega) \to Q_h,\ v \mapsto \Pi^Q(\chi_\Omega v)$ s.t.

$$(\mathcal{E}_h^0 q, r_h)_{\Omega^{\mathcal{T}}} = (q, r_h)_\Omega \qquad \forall r_h \in Q_h.$$

- ➔ $(\operatorname{div} v_h, q)_\Omega = (\operatorname{div} v_h, \mathcal{E}_h^0 q)_{\Omega^{\mathcal{T}}}$
- ➔ $\bar{p}_h \approx \mathcal{E}_h^0 p$
- ➔ $\bar{p}_h \approx p$ on $T \in \mathcal{T}_h^{\text{interior}}$
- ➔ $\bar{p}_h \not\approx p$ on $T \in \mathcal{T}_h^{\text{cut}}$



Backup: Error estimates

Norm on $H(\mathrm{div}, \Omega^{\mathcal{T}})$:

$$\|u\|_{\Sigma}^2 := \|\operatorname{div} u\|_{\Omega^{\mathcal{T}}}^2 + \|u\|_{\Omega_\gamma}^2, \qquad \|u\|_{\Omega_\gamma}^2 := \begin{cases} \|u\|_{\Omega^{\mathcal{T}}}^2 \text{if } \gamma_{\text{👻}} > 0, \\ \|u\|_{\Omega}^2 \text{if } \gamma_{\text{👻}} = 0. \end{cases}$$

Theorem (Error estimate for u_h)

For $u \in H^m(\Omega)$ with $m \in \{0, \ldots, k+1\}$, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} + \|u^{\mathcal{E}} - u_h\|_{\Sigma} + |u_h|_{\text{👻}} \lesssim h^m \|u\|_{H^m(\Omega)} + \mathtt{err}_f.$$

Theorem

For Ω smooth enough to assume L^2-H^2 regularity, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h\|u_h - u\|_{L^2(\Omega)} + h\mathtt{err}_f$$

Backup: Numerical Example

- manufactured solution
- $\mathbb{RT}^k \times \mathbb{P}^k$
- uniform refinements