Ensuring quasi-optimality for the Helmholtz problem

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Model problem & pollution effect

Let Ω be Lipschitz. For a wave-number k, find u s.t.

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Pollution effect[1]**:** For fixed mesh size h, we loose quasi-optimality as the wave-number k increases.

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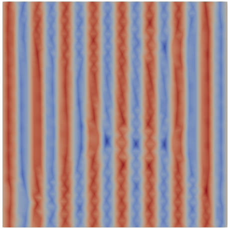
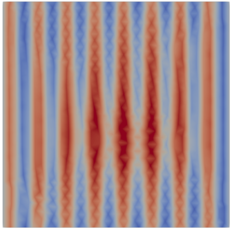


$\Pi_h^{L^2} u_{\mathrm{ex}}$
$h = 0.1$



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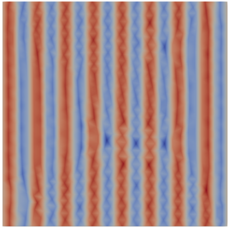

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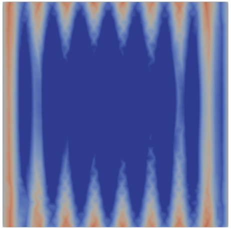

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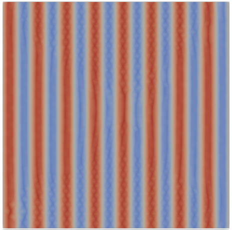

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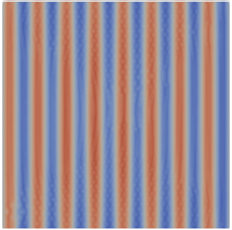

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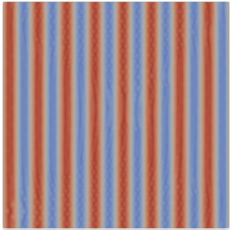

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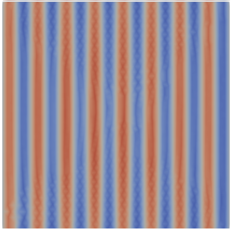

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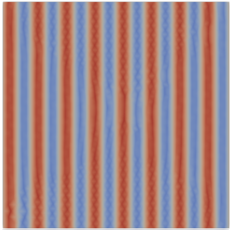

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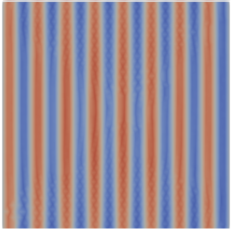

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- ➔ implementable
- ➔ no smoothness assumptions on Ω

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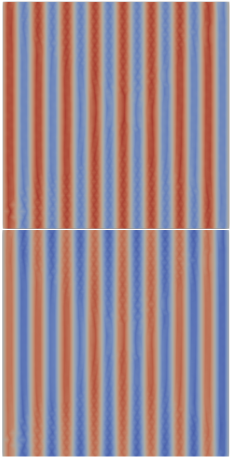

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T-coercivity

Theorem (Ciarlet[2])

Let X be Hilbert and $a(\cdot,\cdot) : X \times X \to \mathbb{C}$ be a bounded sesquilinear form. The problem

$$\textit{find } u \in X \textit{ s.t. } a(u,v) = f(v) \quad \forall v \in X$$

is well-posed iff $\exists T : X \to X$ bijective s.t. $a(\cdot, T\cdot)$ is coercive, i.e.

$$\Re\{a(u,Tu)\} \geq \alpha \|u\|_X^2 \quad \forall u,v \in X.$$

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Note: For Hilbert spaces, T-coercivity is equivalent to the inf sup-condition.

- ➔ necessary & sufficient condition for well-posedness
- ➔ has to be shown on the discrete level (with uniform constant) to conclude quasi-optimality.

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T-coericivity of the Helmholtz problem[3]

Let $(\lambda^{(i)}, e^{(i)})_{i\in\mathbb{N}}$ be the eigenpairs of $-\Delta$ on Ω (normed s.t. $\|e^{(i)}\|_{H^1} = 1$). Define

$$i_* = \max\{i \in \mathbb{N} : \lambda^{(i)} < k^2\} \qquad (\text{assuming } k^2 \notin \{\lambda^{(i)}\})$$

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Then for $u \in H_0^1(\Omega)$

$$\begin{aligned} a(u,u) &:= \int_\Omega \nabla u \cdot \nabla u \, dx - k^2 \int_\Omega u^2 \, dx \\ &= \sum_{i \le i_*} \underbrace{\left(\frac{\lambda^{(i)} - k^2}{1+\lambda^{(i)}}\right)(u^{(i)})^2}_{\le 0} + \sum_{i > i_*} \underbrace{\left(\frac{\lambda^{(i)} - k^2}{1+\lambda^{(i)}}\right)(u^{(i)})^2}_{\ge 0} \\ &\not\ge \alpha \|u\|_{H^1}^2 . \end{aligned}$$

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H^1-conforming discretization

Let $\mathcal{T}_h$ be a triangulation of Ω with mesh size h and $X_h := \mathbb{P}^p(\mathcal{T}_h) \cap H_0^1(\Omega) \subset H_0^1(\Omega)$, then the Galerkin approximation of the Helmholtz problem reads

$$\text{find } u_h \in X_h \text{ s.t. } a(u_h, v_h) = f(v_h) \quad \forall v_h \in X_h.$$

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If $a(\cdot,\cdot)$ is uniformly T_h-coercive, i.e. if $\exists\ T_h : X_h \to X_h$ bijective & $\alpha_* > 0$ independent of h s.t.

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Let $(\lambda_h^{(i)}, e_h^{(i)})_{i\in\mathbb{N}}$ be conforming approximations of $(\lambda^{(i)}, e^{(i)})_{i\in\mathbb{N}}$. Then

- $(e_h^{(i)})$ is a basis of X_h
- $\lambda^{(i)} \leq \lambda_h^{(i)}$ for all i, $\lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \ldots$

(Uniform) Discrete T_h-coercivity

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For $u_h \in X_h$, we have

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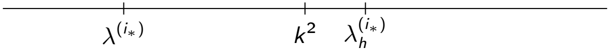
$$\overset{??}{\geq} \alpha \|u_h\|_{H^1}^2$$



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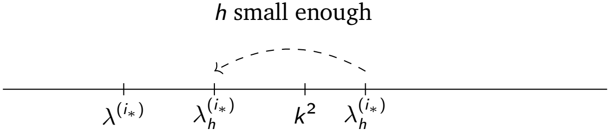
➔ $a(\cdot, \cdot)$ is uniformly T_h-coercive iff $\lambda_h^{(i_*)} < k^2$.



The scheme

➔ We can use this criterion to generate a mesh that guarantees quasi-optimality!

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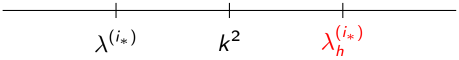

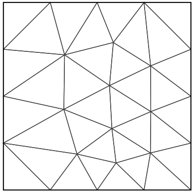

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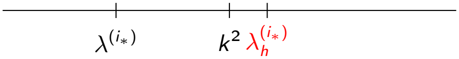

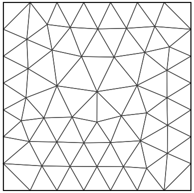

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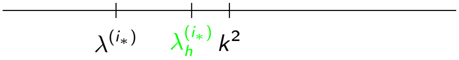

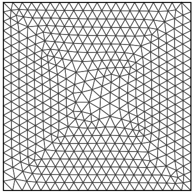

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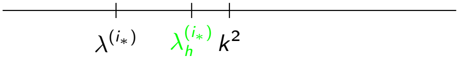

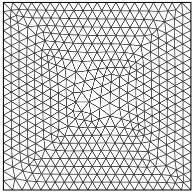

3. Solve the Helmholtz problem on the mesh obtained in Step 2. Since $\lambda_h^{(i_*)} < k^2$, we have quasi-optimality.

Can we optimize the mesh generation process?

- ➔ minimize the number of required mesh elements / dofs

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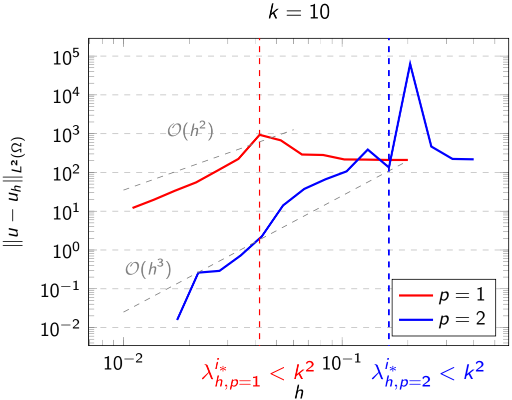
Babuška-Rheinboldt error estimator (averaged over $i_* + \ell$ eigenpairs):

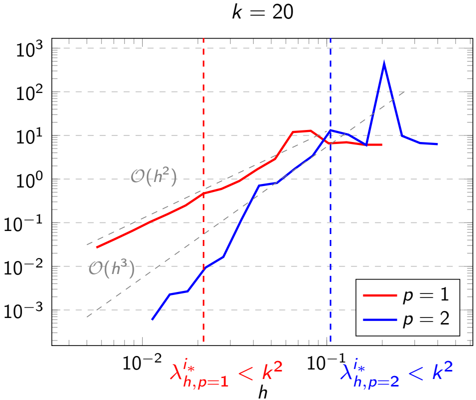
$$\eta = i_*^{-1} \sum_{i=1}^{i_*+\ell} \sum_{K \in \mathcal{T}_h} \left(h_K^2 \|\Delta e_h^{(i)} + \lambda_h^{(i)} e_h^{(i)}\|_{L^2(K)}^2 + \frac{h_K}{2} \|\nabla e_h^{(i)} \cdot n\|_{L^2(\partial K \setminus \partial \Omega)}^2 \right).$$

➔ use adaptive refinements based on this error estimator in Step 2

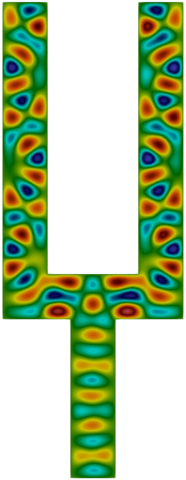
Numerical example - Unit Square

$\Omega = [0,1]^2$, Eigenvalues of $-\Delta$: $\lambda_{i,j} = \pi^2(i^2 + j^2)$, $i, j \in \mathbb{N}$.

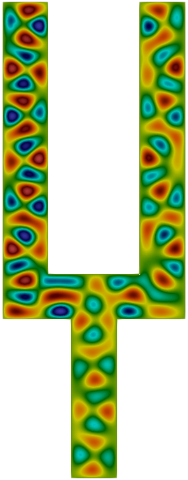




Numerical Example - Tuning fork ($k = 10$)

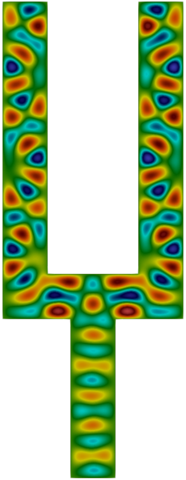


$\lambda_h^{(i_*)} > k^2$

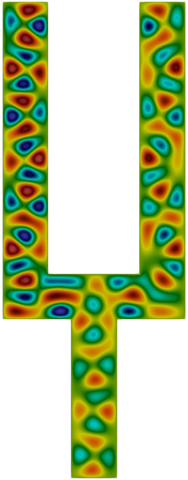


$\lambda_h^{(i_*)} < k^2$

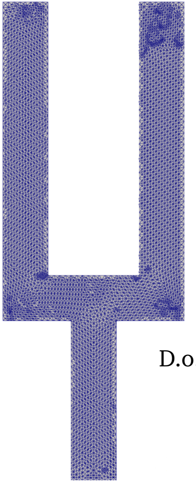
Numerical Example - Tuning fork ($k = 10$)



$\lambda_h^{(i_*)} > k^2$



$\lambda_h^{(i_*)} < k^2$

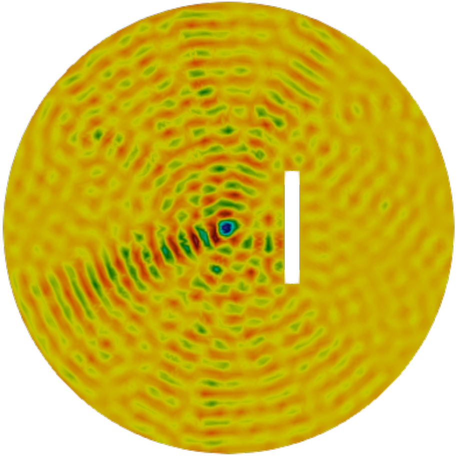


D.o.f.s. required s.t. $\lambda_h^{(i_*)} < k^2$:

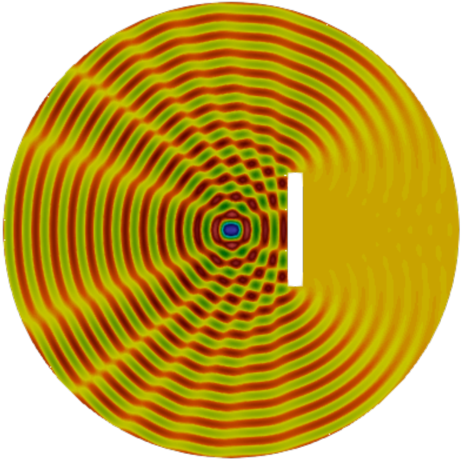
uniform: **336,449**

adaptive: **161,102**

Numerical Example - Scatterer ($k = 100$)



$\lambda_h^{(i_*)} > k^2$



$\lambda_h^{(i_*)} < k^2$

Conclusion

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- quasi-optimality is intimately connected to the discrete eigenvalues of $-\Delta$
- we can use this connection to generate a mesh that guarantees quasi-optimality:
  - ➔ determine maximal index i_* s.t. $\lambda^{(i_*)} < k^2$
  - ➔ adaptively refine the mesh until $\lambda_h^{(i_*)} < k^2$
  - ➔ Solve the Helmholtz problem
- can be extended to Robin / Mixed boundary conditions

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Curious to learn more?

TvB, U. Zerbinati, "An adaptive mesh refinement strategy to ensure quasi-optimality of the conforming finite element method for the Helmholtz equation via T-coercivity" (2024), https://arxiv.org/pdf/2403.06266.