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GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

Pressure-robustness in an axisymmetric setting

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Axisymmetry

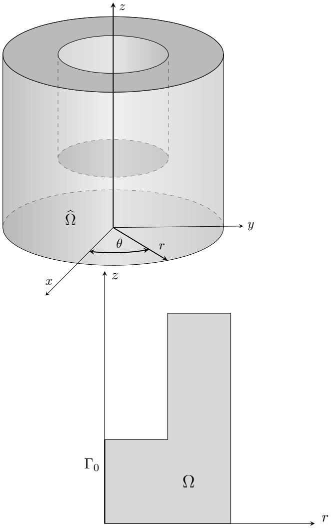
- 3D computations are expensive
- domain & problem axisymmetric ➔ reduced costs
- In cylindrical coordinates (r, θ, z)

$$\widehat{\Omega} = \{(r\cos\theta, r\sin\theta, z) : (r, z) \in \Omega, \theta \in [0, 2\pi]\}$$
$$\Gamma_0 = \Omega \cap \{r = 0\}, \quad \Gamma = \partial\Omega \setminus \Gamma_0$$

- Fourier expansion w.r.t. θ + truncation

$$u(r, \theta, z) = \sum_{k \leq N} u^k(r, z) e^{ik\theta}$$

- axisymmetric data ➔ Solve only for $k = 0$ (for Stokes / Darcy)



Key steps

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- Change of coordinates → transformation of differential operators, e.g.

$$\mathrm{div}_{xyz}\, u = \partial_x u_x + \partial_y u_y + \partial_z u_z, \qquad \mathrm{div}_{\mathrm{axi}}\, u = \partial_r u_r + \frac{1}{r} u_r + \cancel{\frac{1}{r}\partial_\theta u_\theta} + \partial_z u_z$$

- Change of measure: dxdydz $\rightsquigarrow$ $r$drdz → weighted Sobolev-spaces
- BUT: structural properties change, e.g. for the acoustic EVP $\nabla(\mathrm{div}\, \boldsymbol{u}) = \lambda \boldsymbol{u}$, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, discretization with RT- or BDM-elements leads to spurious eigenmodes, in contrast to the 3d case.

J. Querales, R. Rodríguez, P. Venegas, *Num. Approx. of the Displacement Formulation of the Axisymmetric Acoustic Vibration Problem*, SISC, 2021.

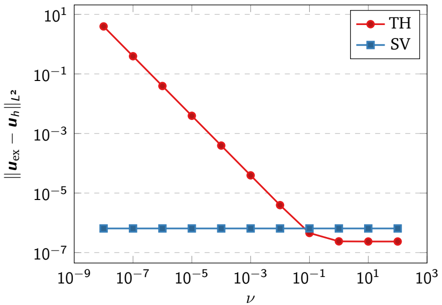
Pressure-Robustness

Given $\boldsymbol{f} \in L^2(\Omega)$, $\nu \in \mathbb{R}_+$, find $(u, p) \in H(\mathrm{div}) \times L^2_0$ s.t.

$$\nu \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f}, \quad \mathrm{div}\, \boldsymbol{u} = 0, \quad \boldsymbol{u}|_{\partial\Omega} = 0$$

- pressure robustness $\widehat{=}$ velocity error independent of pressure error
- exactness of the discrete de Rham complex:

$$\begin{array}{ccc} H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\ \downarrow & & \downarrow \\ \boldsymbol{V}_h & \xrightarrow{\mathrm{div}} & Q_h \end{array}$$



- Taylor-Hood: $(\mathcal{P}^k(\Omega), \mathcal{P}^{k-1}(\Omega))$, $k \geq 2$
- Scott-Vogelius: $(\mathcal{P}^k(\Omega), \mathbb{P}^{k-1}(\mathcal{T}_h))$, $k \geq 4$

V. John, A. Linke, C. Merdon, M. Neilan, L. G. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows.* SIAM Review, 2017.

Pressure-robustness in axisymmetric setting

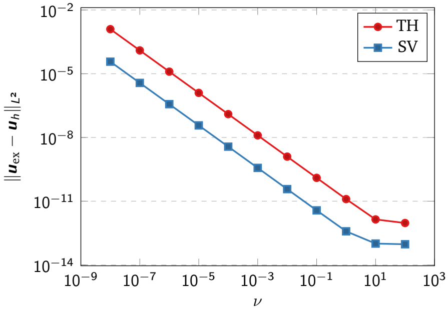
- Change of coordinates transforms the div-operator:

$$\operatorname{div}_{\mathrm{axi}} \boldsymbol{u} = \partial_r \boldsymbol{u}_r + \frac{1}{r}\boldsymbol{u}_r + \partial_z \boldsymbol{u}_z$$

- Problem: $\operatorname{div}_{\mathrm{axi}}$ does not map into $\mathcal{P}^{k-1}$

$$\begin{array}{ccccc} H_{\mathrm{axi}}(\operatorname{div}) & & \xrightarrow{\quad\operatorname{div}_{\mathrm{axi}}\quad} & & L^2_{\mathrm{axi}} \\ \downarrow & & & & \downarrow \\ \boldsymbol{V}_h & \xrightarrow{\operatorname{div}_{\mathrm{axi}}} & \operatorname{div}_{\mathrm{axi}}(\boldsymbol{V}_h) & \xrightarrow{\Pi_h} & Q_h \end{array}$$

V. J. Ervin, *Approximation of axisymmetric darcy flow using mixed finite element methods.* SINUM, 2013.



- Taylor-Hood: $(\mathcal{P}^k(\Omega), \mathcal{P}^{k-1}(\Omega))$, $k \geq 2$
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Weighted Sobolev spaces

- weighted L^2-spaces:

$$\|v\|^2_{L^2_\alpha(\Omega)} := \int_\Omega v^2 r^\alpha \,\mathrm{d}r\,\mathrm{d}z, \quad L^2_\alpha(\Omega) := \{v \text{ measurable} : \|v\|_{L^2_\alpha(\Omega)} < \infty\}$$

- weighted H^1-spaces:

$$\|v\|^2_{H^1_\alpha(\Omega)} := \|\partial_r \partial_z v\|^2_{L^2_\alpha(\Omega)} + \|v\|^2_{L^2_\alpha(\Omega)}, \quad H^1_\alpha(\Omega) := \{v \in L^2_\alpha(\Omega) : \|v\|_{H^1_\alpha(\Omega)} < \infty\}$$

- weighted $H(\mathrm{div})$-spaces:

$$H_\alpha(\mathcal{D}, \Omega) := \{v \in L^2_\alpha(\Omega) : \mathcal{D}v \in L^2_\alpha(\Omega)\}, \quad \mathcal{D} \in \{\mathrm{div}, \mathrm{div}_{\text{axi}}\}.$$

C. Bernardi, M. Dauge, Y. Maday, *Spectral Methods for Axisymm. Domains*. Gauthier-Villars, 1999.

A. Kufner, *Weighted Sobolev Spaces*. Teubner, 1980.

M. Costabel, M. Dauge, J.-Q. Hu, *Characterization of Sobolev spaces by their Fourier coefficients in axisymm. domains*. Calcolo, 2023.

Darcy problem

3D-problem

For given $\boldsymbol{f} \in \boldsymbol{L}^2(\widehat{\Omega})$, $\nu \in \mathbb{R}_+$, find $(\boldsymbol{u}, p)$ s.t.

$$\nu \boldsymbol{u} - \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0, \quad p|_{\partial\Omega} = g$$

Axisymmetric-setting

Find $(\boldsymbol{u}, p) \in H_1(\operatorname{div}_{\mathrm{axi}}) \times L^2_{1,0}$ such that for all $(\boldsymbol{v}, q) \in H_1(\operatorname{div}_{\mathrm{axi}}) \times L^2_{1,0}$:

$$\begin{aligned} \nu \int_\Omega r \boldsymbol{u} \boldsymbol{v} \,\mathrm{d}r\,\mathrm{d}z + \int_\Omega r \operatorname{div}_{\mathrm{axi}} \boldsymbol{v} p \,\mathrm{d}r\,\mathrm{d}z &= \int_\Omega r \boldsymbol{f} \boldsymbol{v} \,\mathrm{d}r\,\mathrm{d}z + \int_\Gamma r g \boldsymbol{v} \cdot \boldsymbol{n} \,\mathrm{d}r\,\mathrm{d}z \\ \int_\Omega r \operatorname{div}_{\mathrm{axi}} \boldsymbol{u} q \,\mathrm{d}r\,\mathrm{d}z &= 0 \end{aligned} \tag{AXI}$$

Modification

Observation: With $\operatorname{div}_{\text{axi}} \boldsymbol{v} = \frac{1}{r}\partial_r(r\boldsymbol{v}_r) + \partial_z \boldsymbol{v}_z$, we have

$$\int_\Omega r \operatorname{div}_{\text{axi}} \boldsymbol{v} p \,\mathrm{d}r\,\mathrm{d}z = \int_\Omega \operatorname{div}(r\boldsymbol{v}) p \,\mathrm{d}r\,\mathrm{d}z$$

→ **Ansatz**: Solve for $\boldsymbol{u} = r\tilde{\boldsymbol{u}}$, i.e. find $(\boldsymbol{u}, p) \in H_{-1}(\operatorname{div}) \times L^2_{1,0}$ s.t.

$$\underbrace{\nu \int_\Omega r^{-1} \boldsymbol{u}\boldsymbol{v} \,\mathrm{d}r\,\mathrm{d}z}_{=:a(\boldsymbol{u},\boldsymbol{v})} + \underbrace{\int_\Omega \operatorname{div} \boldsymbol{v} p \,\mathrm{d}r\,\mathrm{d}z}_{=:b(\boldsymbol{v},p)} = \int_\Omega \boldsymbol{f}\boldsymbol{v} \,\mathrm{d}r\,\mathrm{d}z + \int_\Gamma g\boldsymbol{v}\cdot\boldsymbol{n} \,\mathrm{d}r\,\mathrm{d}z,$$

$$\int_\Omega \operatorname{div} \boldsymbol{u} q \,\mathrm{d}r\,\mathrm{d}z = 0. \qquad \text{(AXI-M)}$$

Equivalence: $(\tilde{\boldsymbol{u}}, p)$ solves (AXI) $\Leftrightarrow$ $(r\tilde{\boldsymbol{u}}, p)$ solves (AXI-M).

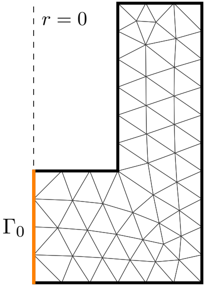
Discrete problem

Remark

If $\boldsymbol{u} \in L^2_{-1}(\Omega)$, then $\boldsymbol{u}|_{\Gamma_0} = 0$.

Discretization with Scott Vogelius elements:

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \mathcal{C}_0(\Omega) \cap L^2_{-1}(\Omega) : \boldsymbol{v}_h|_T \in [\mathcal{P}^k(T)]^2 \; \forall T \in \mathcal{T}_h, \boldsymbol{v}_h|_{\Gamma_0} = 0\},$$
$$Q_h := \mathbb{P}^{k-1}(\mathcal{T}_h) \cap L^2_{1,0}(\Omega)$$



Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$:

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)} + (g, \boldsymbol{v} \cdot \boldsymbol{n})_{L^2(\Gamma)},$$
$$b_h(\boldsymbol{u}_h, q_h) = 0.$$

Stability

For stability, we require continuity and

- $a_h(\cdot,\cdot)$ uniformly coercive on $\ker b_h$, i.e. $a_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \gtrsim \|\boldsymbol{u}_h\|^2_{\boldsymbol{V}_h}$;
- $b_h(\cdot,\cdot)$ fulfills the inf-sup condition:

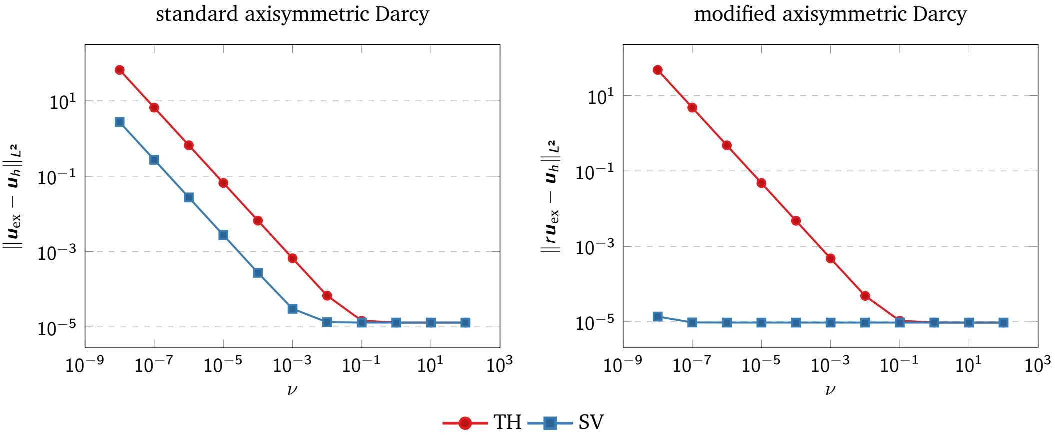
$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \frac{|b_h(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}} \gtrsim \|q_h\|_{Q_h}, \quad \forall q_h \in Q_h.$$

D. Boffi, F. Brezzi, M. Fortin, *Mixed finite element methods and applications*. Springer, 2013.

- choose a "good" (tricky!) norm $\|\cdot\|_{\boldsymbol{V}_h}$, and split $\boldsymbol{V}_h = \ker b_h \oplus \boldsymbol{V}_h^\perp$
- for $k \geq 4$, $\operatorname{div}: \boldsymbol{V}_h \to Q_h$ is surjective (same as 2D case)
- for any $q_h \in Q_h$, choose $\boldsymbol{v}_h^\perp$ s.t. $\operatorname{div} \boldsymbol{v}_h^\perp = \Pi^{k-1}(rq_h)$

$$b_h(\boldsymbol{v}_h,q_h) = \int_\Omega \Pi^{k-1}(rq_h)q_h \,\mathrm{d}r\,\mathrm{d}z = \int_\Omega rq_h^2 \,\mathrm{d}r\,\mathrm{d}z = \|q_h\|^2_{Q_h}$$

- Show that $\|\operatorname{div} \boldsymbol{v}_h^\perp\| \leq \|q_h\|_{Q_h}$


standard axisymmetric Darcy
modified axisymmetric Darcy
$\|\boldsymbol{u}_{\text{ex}} - \boldsymbol{u}_h\|_{L^2}$
$\|r\boldsymbol{u}_{\text{ex}} - \boldsymbol{u}_h\|_{L^2}$
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-9}
10^{-7}
10^{-5}
10^{-3}
10^{-1}
10^{1}
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ν
TH
SV

Outlook

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- Naive axisymmetric discretizations of Stokes / Darcy are not pressure-robust:
  - → even for usually pressure-robust elements (SV, BDM, ...), since $\operatorname{div}_{\mathrm{axi}}(\boldsymbol{V}_h) \not\subset Q_h$
- Solving for $r\boldsymbol{u}$ allows us to work with the usual div-operator
  - → restores pressure robustness for SV
  - → mass-term $\int_\Omega r^{-1} \boldsymbol{u}_h \boldsymbol{v}_h \,\mathrm{d}r\,\mathrm{d}z$ numerically challenging for $r \to 0$
- Details of the analysis still open (e.g. Interpolation?)
- Extension to Stokes??
  - → have to ensure that $\nabla \boldsymbol{u}_h|_{\Gamma_0} = 0$ so that $\int_\Omega r^{-1} \nabla \boldsymbol{u}_h \nabla \boldsymbol{v}_h \,\mathrm{d}r\,\mathrm{d}z$ makes sense

Thank you for your attention!