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Divergence-preserving unfitted FEM for the mixed Poisson problem

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MS50 - Mixed-Dimensional and Immersed Problems:
Numerical Discretization, Analysis and Applications.

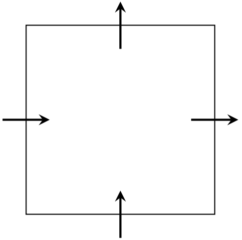
C.L., T.v.B., I.V., *Analysis of div.-preserving unfitted FEM for the mixed Poisson problem.* Math. Comp., 2025.

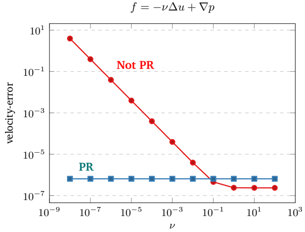
Conservation laws

- physical principles, e.g. conservation of mass:

$$\begin{aligned} \text{continuity eq.:} \quad & \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho \text{ const.:} \quad & \operatorname{div} u = 0 \end{aligned}$$

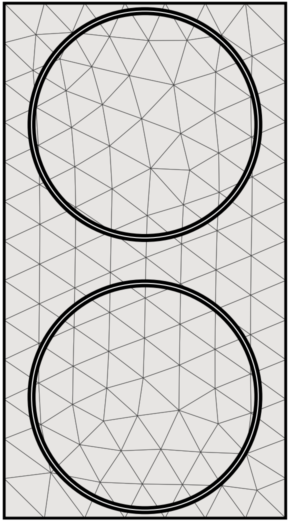
- desirable to transfer this to the discrete level, e.g. for *pressure robustness* of the Stokes problem
- How to obtain divergence-preservation in an unfitted setting?





Unfitted FEM

- How to handle complicated (moving) geometries, possibly with topology changes?
- Unfitted FEM / Cut FEM / XFEM / Immersed FEM: separate the geometry description from the computational domain
- Challenges:
  - Imposition of boundary conditions
  - Numerical integration (robustness / high order?)
  - Stability w.r.t. small cuts



E. Burman, P. Hansbo, M. G. Larson, S. Zahedi, *Cut finite element methods.* Acta Numerica, 2025.

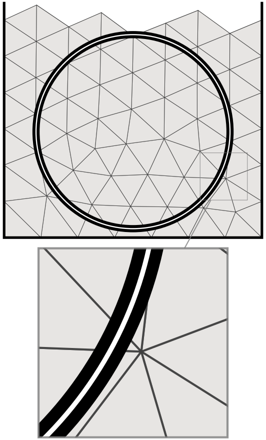
C. Lehrenfeld, M. A. Olshanskii, *An Eulerian FEM for PDEs in time-dependent domains.* ESAIM: M2AN, 2019.

Stability wrt small cuts

- need to obtain stability with respect to small cuts
- Remedy: ghost penalty stabilization

$$\text{👻}(u_h, v_h) := \sum_{F \in \mathcal{F}_h^{\partial\Omega}} \sum_{l=0}^{k} \frac{h_F^{2l+1}}{l!^2} \int_F [\![\partial_n^l u_h]\!][\![\partial_n^l v_h]\!] ds$$

- Grants stability through: $\|u\|_{H^q(\Omega)} + |u|_{\text{👻}} \simeq \|u\|_{H^q(\Omega^{\mathcal{T}})}$



E. Burman, *Ghost penalty.* C.R. Math., 2010.

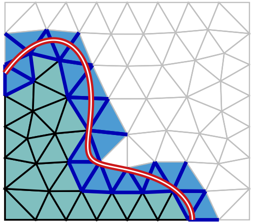
J. Preuss, *Higher order unfitted isoparametric space-time FEM on moving domains.* 2018.

Mass-conserving unfitted FEM?

Model problem: Mixed Poisson / Darcy

Find u, p with $p = p_D$ on $\partial\Omega$ such that

$$u - \nabla p = 0 \text{ in } \Omega,$$
$$\operatorname{div} u = -f \text{ in } \Omega.$$



▷ $T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma$

▷ $T \in \mathcal{T}_h^\Gamma$

$\Gamma = \partial\Omega$

▷ $\cup$ ▷ $= \Omega^{\mathcal{T}}$

Naive variational formulation (unfitted)

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h) \subset H(\operatorname{div}, \Omega^{\mathcal{T}})$ and $p_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h) \subset L^2(\Omega^{\mathcal{T}})$ s.t.

$$\begin{aligned} (u_h, v_h)_{L^2(\Omega)} + \text{👻} + (\operatorname{div} v_h, p_h)_{L^2(\Omega)} + \text{👻} &= (v_h, p_D)_{L^2(\partial\Omega)} \qquad && \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_{L^2(\Omega)} + \text{👻} &= (-f, q_h)_{L^2(\Omega)} && \forall q_h \in Q_h. \end{aligned}$$

→ Stable, but mass balance is polluted by 👻

Current literature

Stab.	Cons.	HO.	
●	○	●	R. Puppi, *A cut FEM for the* ***Darcy*** *problem.* arXiv, 2021.
●	○	●	P. Cao, J. Chen, *An extended FEM for coupled* ***Darcy-Stokes*** *problems.* IJNME, 2022.
●	●	●	T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A Divergence Preserving Cut FEM for* ***Darcy*** *Flow.* SISC, 2024.
●	●	●	T. Frachon, E. Nilsson, S. Zahedi, *Divergence-free cut FEMs for* ***Stokes*** *flow.* BIT, 2024.
●	●	○	E. Burman, P. Hansbo, M. G. Larson, *Cut FEM for Divergence-Free Approx. of* ***Incompr. Flow:*** *A Lagrange Multiplier Approach.* SINUM., 2024.
●	●	●	C. Lehrenfeld, TvB, I. Voulis, *Analysis of div.-preserving unfitted FEM for the mixed poisson problem.* Math. Comp., 2025.

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Stability of saddle point problem

Set $a(u_h, v_h) := (u_h, v_h)_{L^2(\Omega)}$ and $b(v_h, p_h) = (\operatorname{div} v_h, p_h)_{L^2(\Omega)}$.

- Kernel coercivity of $a(\cdot,\cdot)$:

$$a(u_h, u_h) \geq \alpha \|u_h\|^2_{\Sigma_h} \quad \forall u_h \in \Sigma_h \cap \ker B_h$$

- Inf-sup stability of $b(\cdot,\cdot)$:

$$\inf_{q_h \in Q_h} \sup_{v_h \in \Sigma_h} \frac{|b(v_h, q_h)|}{\|v_h\|_{\Sigma_h} \|q_h\|_{Q_h}} \geq \gamma > 0.$$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}$$

→ Modify $b(\cdot,\cdot)$ without changing its kernel?

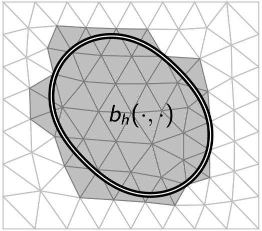
D. Boffi, F. Brezzi, M. Fortin, *Mixed FEM and Applications*. Springer, 2013.

Approaches

Goal: Modify $b(v_h, p_h) = (\operatorname{div} v_h, p_h)_{L^2(\Omega)}$ to obtain stability without polluting the mass balance

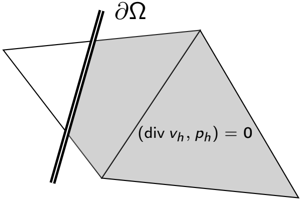
- Extend domain[1,2] of $b(\cdot,\cdot)$: $\Omega \rightsquigarrow \Omega^{\mathcal{T}}$

$$b_h(v_h, p_h) := (\operatorname{div} v_h, p_h)_{\Omega^{\mathcal{T}}}$$
$$\Rightarrow \ker b_h = \ker b$$



- Modify 👻$(\cdot,\cdot)^3$:

$$b_h(v_h, p_h) = (\operatorname{div} v_h, p_h)_{L^2(\Omega)} + \gamma_{\text{👻}}\text{👻}(\operatorname{div} v_h, p_h)$$



1. Burman, Hansbo, Larson, 2024; 2. Lehrenfeld, vB, Voulis, 2025
3. Frachon, Nilsson, Zahedi, 2024

Div.-preserving unfitted mixed FEM

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Stable unfitted mixed Poisson

With $\gamma_{\text{👻}} \geq 0$, find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$ and $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$ such that

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, \bar{p}_h) &= (v_h \cdot n, p_D)_{L^2(\partial\Omega)} && \forall v_h \in \Sigma_h, \\ b_h(u_h, q_h) &= -({\color{red}f_h}, q_h)_{L^2({\color{red}\Omega^{\mathcal{T}}})} && \forall q_h \in Q_h, \end{aligned} \tag{1}$$

with the bilinear forms

$$a_h(u_h, v_h) := a(u_h, v_h) + \gamma_{\text{👻}} \text{👻}(u_h, v_h); \qquad b_h(v_h, p_h) := (\operatorname{div} v_h, p_h)_{L^2(\Omega^{\mathcal{T}})}.$$

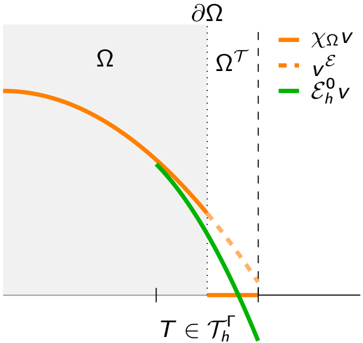
- f_h is a suitable discrete extension of f from Ω to $\Omega^{\mathcal{T}}$ (Assump.: approx. $f^{\mathcal{E}}$ well)
- (1) is stable (independent of cut position), u_h consistent with u (up to $f_h \approx f$)
- Mass balance is preserved

Interpretation of $\bar{p}_h$

- $\bar{p}_h \approx p$ on $T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma$, but $\bar{p}_h \not\approx p$ on cut elements $T \in \mathcal{T}_h^\Gamma$
- Define $\mathcal{E}_h^0 : L^2(\Omega) \to Q_h$, $v \mapsto \Pi_{Q_h}(\chi_\Omega \cdot v)$ s.t.

$$(\mathcal{E}_h^0 q, r_h)_{L^2(\Omega^\mathcal{T})} = (\chi_\Omega \cdot q, r_h)_\Omega \quad \forall r_h \in Q_h,$$

→ $\mathcal{E}_h^0$ is the $L^2(\Omega^\mathcal{T})$-projection into Q_h of the extension by zero



Interpretation of $\bar{p}_h$

It holds $b(v_h, q) = (\mathrm{div}\, v_h, q)_{L^2(\Omega)} = (\mathrm{div}\, v_h, \mathcal{E}_h^0 q)_{\Omega^\mathcal{T}} = b_h(v_h, \mathcal{E}_h^0 q)$ and thus

$$\bar{p}_h \approx \mathcal{E}_h^0 p.$$

Error estimates

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Norm on $H(\operatorname{div}, \Omega^{\mathcal{T}})$:

$$\|u\|_{\Sigma}^2 := \|\operatorname{div} u\|_{\Omega^{\mathcal{T}}}^2 + \|u\|_{\Omega_\gamma}^2, \qquad \|u\|_{\Omega_\gamma}^2 := \begin{cases} \|u\|_{L^2(\Omega^{\mathcal{T}})}^2 & \text{if } \gamma_{\text{👻}} > 0, \\ \|u\|_{L^2(\Omega)}^2 & \text{if } \gamma_{\text{👻}} = 0. \end{cases}$$

$p^{\mathcal{E}}$ smooth Sobolev extension of p, $u^{\mathcal{E}} = \nabla p^{\mathcal{E}}$, $f^{\mathcal{E}} = -\operatorname{div} u^{\mathcal{E}}$.

Error estimate for u_h

For $u \in H^m(\Omega)$ with $m \in \{0, \dots, k+1\}$, there holds for $\gamma_{\text{👻}} \geq 0$

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} + \|u^{\mathcal{E}} - u_h\|_{\Omega_\gamma} + \gamma_{\text{👻}}^{1/2} |u_h|_{\text{👻}} \lesssim h^m \|u\|_{H^m(\Omega)} + \|\Pi^Q f^{\mathcal{E}} - f_h\|_{\Omega^{\mathcal{T}}}.$$

Theorem

For Ω smooth enough to assume L^2-H^2 regularity, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h \left(\|u_h - u\|_{L^2(\Omega)} + \gamma_{\text{👻}}^{1/2} |u|_{\text{👻}} \right) + \|f - f_h\|_{-2}$$

Error estimates

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Norm on $H(\mathrm{div}, \Omega^{\mathcal{T}})$:

$$\|u\|_{\Sigma}^2 := \|\,\mathrm{div}\, u\|_{\Omega^{\mathcal{T}}}^2 + \|u\|_{\Omega_\gamma}^2, \qquad \|u\|_{\Omega_\gamma}^2 := \begin{cases} \|u\|_{L^2(\Omega^{\mathcal{T}})}^2 & \text{if } \gamma_{\text{👻}} > 0, \\ \|u\|_{L^2(\Omega)}^2 & \text{if } \gamma_{\text{👻}} = 0. \end{cases}$$

$p^{\mathcal{E}}$ smooth Sobolev extension of p, $u^{\mathcal{E}} = \nabla p^{\mathcal{E}}$, $f^{\mathcal{E}} = -\,\mathrm{div}\, u^{\mathcal{E}}$.

Theorem

For Ω smooth enough to assume L^2-H^2 regularity, there holds for the solutions $(u, p) \in H^{\ell+1}(\Omega) \times H^{\ell}(\Omega)$, $f \in H^{\ell+1}(\Omega)$, $0 \leq \ell \leq k$ and $f_h \approx f^{\mathcal{E}}$ well-enough that

$$\|u^{\mathcal{E}} - u_h\|_{\Sigma} + \gamma_{\text{👻}}^{1/2} |u_h|_{\text{👻}} \lesssim h^{\ell+1}\|u\|_{H^{\ell+1}(\Omega)} + h^{\ell+1}\|f\|_{H^{\ell+1}(\Omega)},$$

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h^{\ell+2}\|u\|_{H^{\ell+1}(\Omega)} + h^{\ell+2}\|f\|_{H^{\ell+1}(\Omega)}.$$

Recall the mixed Poisson problem:

$$u - \nabla p = 0 \text{ in } \Omega, \quad \operatorname{div} u = -f \text{ in } \Omega, \quad p = p_D \text{ on } \partial\Omega$$

We solve for $u_h \in \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in \mathbb{P}^k(\mathcal{T}_h)$

- We can use the relationship $\nabla p = u$ for a *post-processing* strategy
- Two goals:
  1. repair the inconsistency: $\bar{p}_h \not\approx p$ on $\mathcal{T}_h^\Gamma$
  2. additional order of accuracy for $\bar{p}_h \approx p$

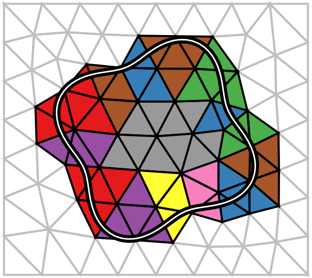
Post-processing

Recall the mixed Poisson problem:

$$u - \nabla p = 0 \text{ in } \Omega, \quad \operatorname{div} u = -f \text{ in } \Omega, \quad p = p_D \text{ on } \partial\Omega$$

We solve for $u_h \in \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in \mathbb{P}^k(\mathcal{T}_h)$

- We can use the relationship $\nabla p = u$ for a *post-processing* strategy
- Two goals:
  1. repair the inconsistency: $\bar{p}_h \not\approx p$ on $\mathcal{T}_h^\Gamma$
  2. additional order of accuracy for $\bar{p}_h \approx p$
- Two versions:
  1. Elementwise
  2. Patchwise

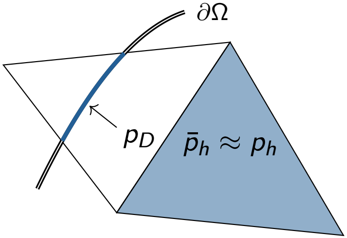


Elementwise Post-processing

Element-local Scheme

For each $T \in \mathcal{T}_h$, find $p_h^* \in \mathbb{P}^{k+1}(T)$ s.t.

$$\begin{aligned}
(\nabla p_h^*, \nabla q_h^*)_T &= (u_h, \nabla q_h^*)_T && \forall q_h^* \in \mathbb{P}^{k+1}(T) \setminus \mathbb{R}, \\
(p_h^*, 1)_T &= (\bar{p}_h, 1)_T && \text{if } T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma, \\
(p_h^*, 1)_{T\cap\partial\Omega} &= (p_D, 1)_{T\cap\partial\Omega} && \text{if } T \in \mathcal{T}_h^\Gamma.
\end{aligned}$$



Error estimate

If $\gamma_{\text{👻}} > 0$ and $p^{\mathcal{E}} \in H^{k+2}(\Omega^{\mathcal{T}})$, Ω smooth enough s.t. L^2-H^2 regularity holds, $f \in H^{k+1}(\Omega)$, then

$$\|p^{\mathcal{E}} - p_h^*\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h^{k+2}(\|p\|_{H^{k+2}(\Omega)} + \|f\|_{H^{k+1}(\Omega)}).$$

→ depends on Dirichlet data & requires $\gamma_{\text{👻}} > 0$

Patchwise Post-processing

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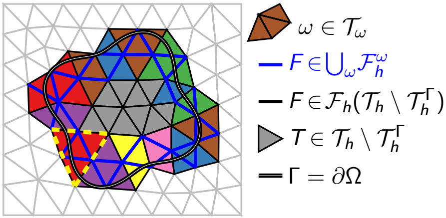
Patchwise Scheme

For each $\omega \in \mathcal{T}_\omega$, find $p_h^* \in \mathbb{P}^{k+1}(\omega)$ s.t. for all $q_h^* \in \mathbb{P}^{k+1}(\omega) \setminus \mathbb{R}$:

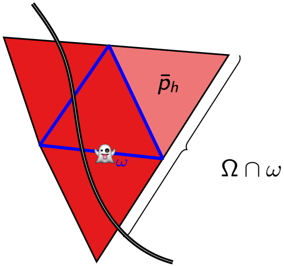
$$(\nabla p_h^*, \nabla q_h^*)_{\Omega\cap\omega} + \text{👻}_\omega(p_h^*, q_h^*) = (u_h, \nabla q_h^*)_{\Omega\cap\omega},$$

$$(p_h^*, 1)_{\Omega^{\mathrm{int}}\cap\omega} = (\bar{p}_h, 1)_{\Omega^{\mathrm{int}}\cap\omega}.$$



Error estimate

For $p \in H^{k+2}(\Omega)$, Ω smooth enough (L^2-H^2 reg.), $f \in H^{k+1}(\Omega)$, it holds $\|p^{\mathcal{E}} - p_h^*\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h^{k+2}(\|p\|_{H^{k+2}(\Omega)} + \|f\|_{H^{k+1}(\Omega)})$.



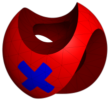
- → no dependence on Dirichlet boundary data
- → $\gamma_{\text{👻}} = 0$ allowed (hybridization possible)

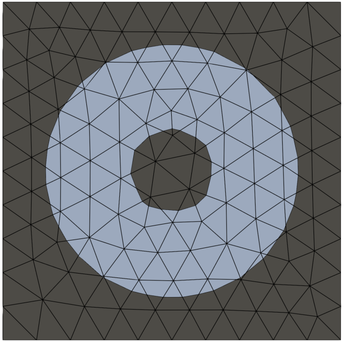
Numerical examples

- manufactured solution:

$$p = \sin(x), u = \nabla p, f = -\Delta p$$

- ring geometry via levelset
- isoparametric unfitted FEM
- $\mathbb{RT}^k \times \mathbb{P}^k$, uniform refinements
- implementation with `ngsxfem`

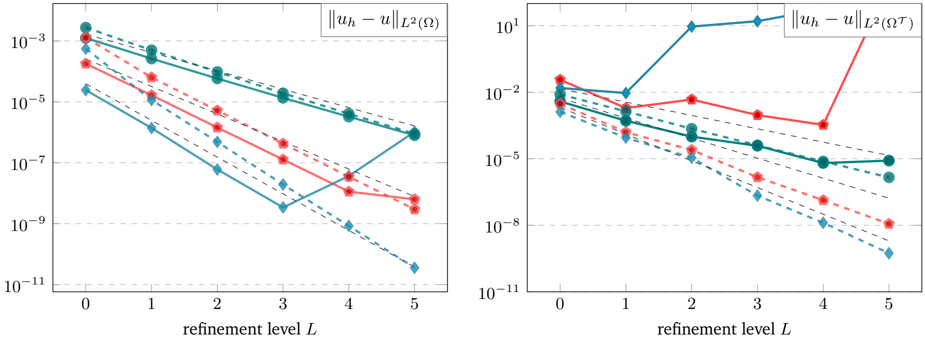




C. Lehrenfeld, *High order unf. FEMs on level set doms. using isoparametric mappings*. CMAME, 2016.

C. Lehrenfeld, F. Heimann, J. Preuss, H. von Wahl, *ngsxfem: Add-on to NGSolve for geometrically unfitted finite element discretization*. JOSS, 2021.

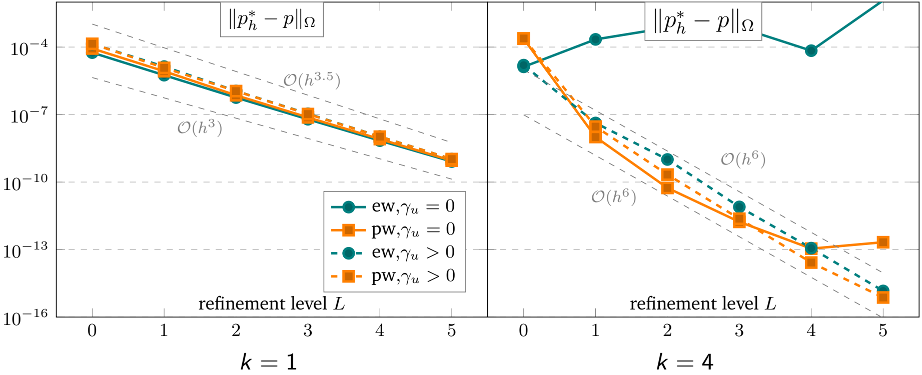
Numerical examples: Convergence of $u - u_h$



$\gamma_{👻} > 0$: dashed, $\gamma_{👻} = 0$ (solid), $k \in \{1, 2, 3\}$

- for accuracy on $\Omega^\mathcal{T}$, $\gamma_{👻} > 0$ is necessary
- for accuracy on Ω, $\gamma_{👻} = 0$ is possible but conditioning issues possible

Numerical examples: Post-processing



- convergence of order $\mathcal{O}(h^{k+2})$ (as expected)
- element-wise post-processing requires $\gamma_{👻} > 0$

Conclusion

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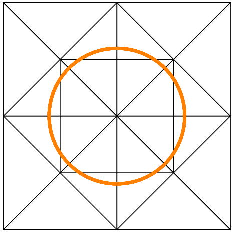
Divergence-preserving unfitted FEM

- Adding standard 👻-penalty to $(\operatorname{div} u_h, q_h)_{L^2(\Omega)}$ pollutes the mass balance
- Extension from Ω to $\Omega^{\mathcal{T}}$ yields inf-sup stability (independent of cut), mass conservation & consistent approx. of u
- $\bar{p}_h \not\approx p$ on $\mathcal{T}_h^{\Gamma}$, but can be repaired with post-processing:
  - Elementwise: requires $\gamma_{👻} > 0$ & Dirichlet data;
  - Patchwise: $\gamma_{👻} = 0$ possible, independent of Dirichlet data
- $\gamma_{👻} = 0$ possible (➔ hybridization), but conditioning issues possible

Thank you for your attention!

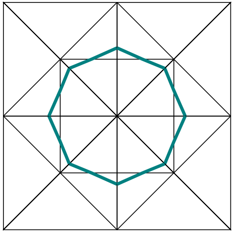
C. Lehrenfeld, TvB, I. Voulis, *Analysis of divergence-preserving unfitted finite element methods for the mixed Poisson problem.* Math. Comp., 2025.

Backup: Isoparametric mapping



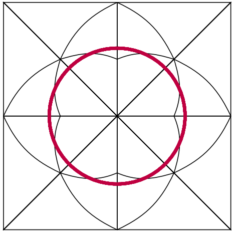
$\{\phi_h = 0\}$
implicit, higher order

$+$



$\{\hat{\phi}_h = 0\}$
explicit, only 2nd order

$\xrightarrow{\Theta_h}$



$\Theta_h(\{\hat{\phi}_h = 0\})$
explicit, higher order

Construct parametric mapping Θ_h of underlying mesh such that $\hat{\phi}_h \approx \phi_h \circ \Theta_h$:

$$\rightsquigarrow \operatorname{dist}\left(\Gamma_h, \partial\left(\Theta_h(\Gamma^{\text{lin}})\right)\right) \leq \mathcal{O}(h^{k_s+1}).$$

Allows to work with $\{\hat{\phi}_h = 0\}$ as reference and guarantees robust quadrature.

C. Lehrenfeld, *High order unf. FEMs on level set doms. using isoparametric mappings*. CMAME, 2016.

Backup: Assumptions on f_h

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- Fitted FEM: $f_h = \Pi^Q f = -\Pi^Q \operatorname{div} u = -\operatorname{div} u_h$, but here $-\operatorname{div} u_h = f_h \neq \Pi^Q f^{\mathcal{E}}$
- Assumption: f_h approximates $f^{\mathcal{E}} := -\operatorname{div} u^{\mathcal{E}}$ well-enough s.t. for $r \in \{0, \dots, k_f + 1\}$, $k_f \in \mathbb{N}$

$$\|f_h - f^{\mathcal{E}}\|_{L^2(\Omega^{\mathcal{T}})} + h^{-1}\|f_h - f\|_{-2} \lesssim h^r \|f\|_{H^r(\Omega)},$$

with $\|\cdot\|_{-2} \widehat{=}$ operator norm over functionals $H^2(\Omega) \cap H^1_0(\Omega)$.

Possible choice of f_h

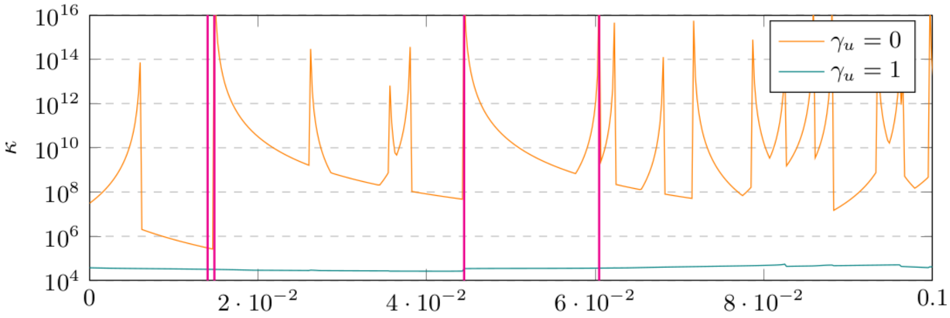
For $\gamma_f > 0$, find $f_h \in \mathbb{P}^{k_f}(\mathcal{T}_h)$ s.t.

$$(f_h, q_h)_{L^2(\Omega)} + \gamma_f \text{👻}(f_h, q_h) = (f, q_h)_{L^2(\Omega)} \qquad \forall q_h \in \mathbb{P}^{k_f}(\mathcal{T}_h).$$

→ fulfills the assumptions from above!

- analysis requires $k_f \geq k$, but numerics indicate that this could be weakened to $k_f = k - 1$

Backup: Conditioning (Numex)



- $\gamma_{👻} > 0$ ensures that the condition number is bounded