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A C^0-hybrid IP method for the nematic Helmholtz–Korteweg equation

Tim van Beeck[1]

Based on previous work with: Patrick E. Farrell[2,3] & Umberto Zerbinati[2]

[1]Institute for Numerical and Applied Mathematics, University of Göttingen
[2]Mathematical Institute, University of Oxford
[3]Mathematical Institute, Charles University, Prague

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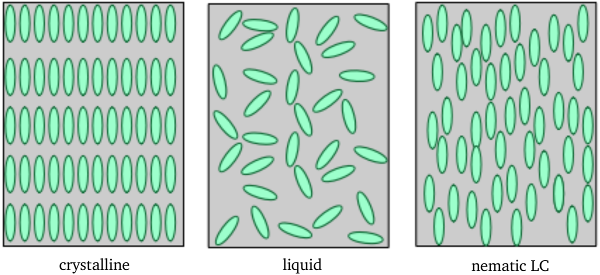
Nematic LCs

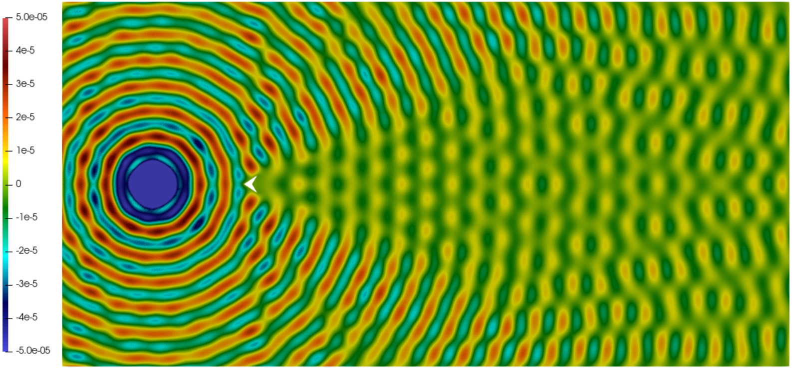
- Liquid crystals (LCs): fluids with ordered molecules
- Common example: Liquid crystal displays (**LCDs**), but also many **biological** systems
- nematic LC: molecules have common alignment



Our objective: study acoustic wave propagation in nematic LCs

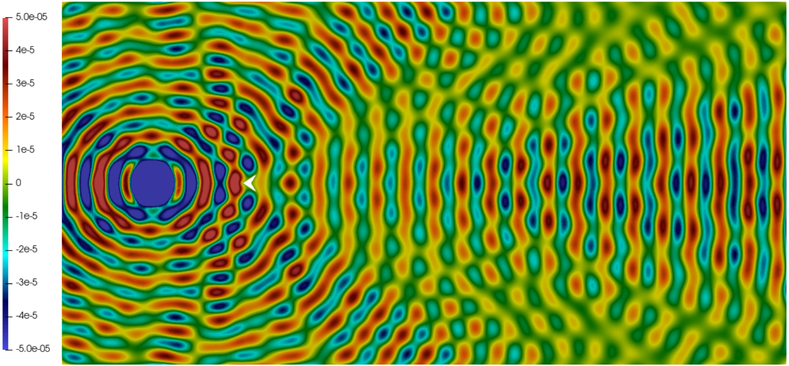
Appetizer

► Scattering by sound-soft obstacle: no nematic field



Appetizer

- Scattering by sound-soft obstacle: $\boldsymbol{n}_x = 1.0, \boldsymbol{n}_y = 0.0$ if $x \leq 0.5, -1.0$ else



Helmholtz–Korteweg equation

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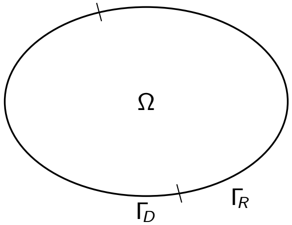
Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{C}$, $d = 2, 3$, such that

$$\alpha \Delta^2 u + \beta \nabla \cdot \nabla \left(\boldsymbol{n}^\top (D^2 u) \boldsymbol{n} \right) - \Delta u - k^2 u = f \text{ in } \Omega,$$
$$\mathcal{B}_D u = 0 \text{ on } \Gamma_D,$$
$$\mathcal{B}_R u = 0 \text{ on } \Gamma_R,$$



- $\Gamma_D \cup \Gamma_R = \partial\Omega$, $\Gamma_D \cap \Gamma_R = \emptyset$, $\Gamma_D = \emptyset$, $\Gamma_R = \emptyset$ allowed
- sound soft boundary conditions

$$\mathcal{B}_D u := \left(u, \alpha \Delta u + \beta \boldsymbol{n}^\top (D^2 u) \boldsymbol{n}\right)$$

- impedance boundary conditions

$$\mathcal{B}_R u := \left(\partial_n u - i\theta u, \alpha \partial_n \Delta u - i\theta(\alpha \Delta u + \beta \boldsymbol{n}^\top (D^2 u) \boldsymbol{n} - \beta \partial_n (\boldsymbol{n}^\top (D^2 u) \boldsymbol{n}))\right)$$

Farrell, Zerbinati, *Time-harmonic waves in Korteweg & nematic-Korteweg fluids.* PRE, 2025.

Weak formulation & conforming discretization

- Weak formulation: For $\epsilon > 0$, find $u \in V := \{v \in H^{2+\epsilon}(\Omega) : v|_{\Gamma_D} = 0\}$ s.t.

$$a(u, v) = (f, v)_\Omega \qquad \forall v \in V, \quad \text{where}$$

$$a(u, v) := \alpha(\Delta u, \Delta v)_\Omega + \beta(\boldsymbol{n}^\top (D^2 u)\boldsymbol{n}, \Delta v)_\Omega + (\nabla u, \nabla v)_\Omega - k^2(u, v)_\Omega + c^{\Gamma_R}(u, v).$$

- $c^{\Gamma_R}(\cdot, \cdot)$ encodes impedance boundary terms
- $V_h \subset V$ requires H^2-conformity ➔ C^1-conforming elements, e.g. Argyris, ...
  - implementation tricky (especially in 3D)
  - polynomial degree needs to be suff. high (e.g. $p \geq 5$ (9) in 2D (3D) for Argyris)
  - usually requires Nitsche for Dirichlet boundary conditions
  - singularly perturbed problem ➔ Morley might fail (not conv. for 2nd order)
- Here: non-conforming ansatz

Farrell, vB, Zerbinati, *Analysis & num. analysis of the nem. Helmholtz–Korteweg eq.* arXiv, 2025.

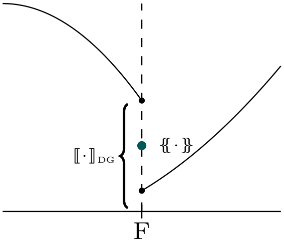
C^0-interior penalty method

- Fully non-conforming ansatz (DG):
  $u_h, v_h \in V_h := \mathbb{P}^p(\mathcal{T}_h) \subset L^2(\Omega)$

$$(\Delta^2 u_h, v_h)_\Omega = \sum_{T\in\mathcal{T}_h} (\Delta u_h, \Delta v_h)_T$$

$$+ \sum_{F\in\mathcal{F}_h} ((\{\!\{\partial_\nu(\Delta u_h)\}\!\}, [\![v_h]\!]_{\mathrm{DG}})_F + (\{\!\{\Delta u_h\}\!\}, [\![\partial_\nu u_h]\!]_{\mathrm{DG}})_F)$$

$+$ symmetry terms (2x) $+$ stabilitzation terms (2x)



- C^0-interior penalty method: $u_h, v_h \in \mathbb{P}^p(\mathcal{T}_h) \cap H^1(\Omega) \not\subset H^2(\Omega)$

$$(\Delta^2 u_h, v_h)_\Omega = \sum_{T\in\mathcal{T}_h} (\Delta u_h, \Delta v_h)_T + \sum_{F\in\mathcal{F}_h} (\underbrace{(\{\!\{\partial_\nu(\Delta u_h)\}\!\}, [\![v_h]\!]_{\mathrm{DG}})_F}_{=0} + (\{\!\{\Delta u_h\}\!\}, [\![\partial_\nu u_h]\!]_{\mathrm{DG}})_F)$$

Süli, Mozolevski, *hp-version of IP DGFEMs for the biharmonic equation.* CMAME, 2007.

Brenner, Sung, *C0-IP methods for 4th order elliptic BVPs on polygonal domains.* JSC, 2005.

Hybridization

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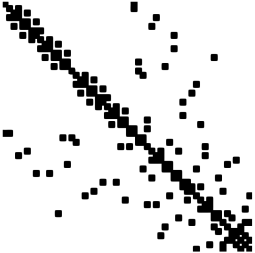
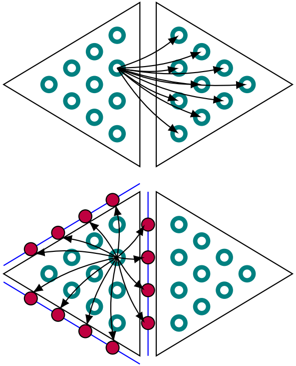
- Reduce coupling by introducing facet unknowns





- Static condensation: Use the Schur complement with $S = A_{\mathcal{T}_n\mathcal{T}_n} - A_{\mathcal{T}_n\mathcal{F}_n}A^{-1}_{\mathcal{F}_n\mathcal{F}_n}A_{\mathcal{F}_n\mathcal{T}_n}$

$$\begin{pmatrix} A_{\mathcal{T}_n\mathcal{T}_n} & A_{\mathcal{T}_n\mathcal{F}_n} \\ A_{\mathcal{F}_n\mathcal{T}_n} & A_{\mathcal{F}_n\mathcal{F}_n} \end{pmatrix} = \begin{pmatrix} I & A_{\mathcal{T}_n\mathcal{F}_n}A^{-1}_{\mathcal{F}_n\mathcal{F}_n} \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} S & 0 \\ 0 & A_{\mathcal{F}_n\mathcal{F}_n} \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ A^{-1}_{\mathcal{F}_n\mathcal{F}_n}A_{\mathcal{T}_n\mathcal{F}_n} & I \end{pmatrix}$$

C^0-hybrid interior penalty method

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- Approximation space: $V_h := \mathbb{P}^p(\mathcal{T}_h) \cap H^1(\Omega) \times \mathbb{P}^{p-1}(\mathcal{F}_h) \ni u_h = (u_\mathcal{T}, \sigma_F)$
  $\mathbb{P}^p(\mathcal{T}_h) \cap H^1(\Omega) \subset H^1(\Omega)$: Lagr. polyn., $\mathbb{P}^{p-1}(\mathcal{F}_h)$: disc. polyn. on $\mathcal{F}_h$,
  $u_\mathcal{T} \approx u$, $\sigma_F \approx \partial_\nu u|_{\mathcal{F}_h}$
- HDG-jump operator: $[\![\underline{\partial_\nu u_h}]\!] = \partial_\nu u_\mathcal{T} - \sigma_F$
- With $\partial\mathcal{T}_h = \mathcal{F}_h^{\text{int}} \bigcup (\partial\mathcal{T}_h \cap \partial\Omega)$ and $u \in V$ s.t. $[\![\Delta u]\!]_{\text{DG}} = 0$, we have

$$(\Delta u, \mu_F)_{\partial\mathcal{T}_h} = ([\![\Delta u]\!]_{\text{DG}}, \mu_F)_{\mathcal{F}_h^{\text{int}}} + (\Delta u, \mu_F)_{\partial\mathcal{T}_h \cap \partial\Omega} = (\Delta u, \mu_F)_{\mathcal{F}_h^R},$$

- Then, we have that

$$(\Delta^2 u_\mathcal{T}, v_\mathcal{T})_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (\Delta u_\mathcal{T}, \Delta v_\mathcal{T})_T - (\Delta u_\mathcal{T}, [\![\underline{\partial_\nu v_h}]\!])_{\partial T} - \sum_{F \in \mathcal{F}_h \cap \Gamma_R} (\Delta u_\mathcal{T}, \mu_F)_F$$

Discrete problem

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Find $u_h = (u_\mathcal{T}, \sigma_F) \in V_h := \mathbb{P}^p(\mathcal{T}_h) \cap H^1(\Omega) \times \mathbb{P}^{p-1}(\mathcal{F}_h)$ s.t.

$$a_h(u_h, v_h) = (f, v_h)_\Omega \qquad \forall v_h = (v_\mathcal{T}, \mu_F) \in V_h,$$

where $a_h(u_h, v_h) = \alpha a_h^\Delta(u_h, v_h) + \beta a_h^{D^2}(u_h, v_h) + (\nabla u_\mathcal{T}, \nabla v_\mathcal{T}) - k^2(u_\mathcal{T}, v_\mathcal{T}) + c_h^{\Gamma_R}(u_h, v_h)$

$$\begin{aligned}
a_h^\Delta(u_h, v_h) := \sum_{T\in\mathcal{T}_h} & (\Delta u_\mathcal{T}, \Delta v_\mathcal{T})_T - (\Delta u_\mathcal{T}, [\![\underline{\partial_{\boldsymbol{\nu}} v_h}]\!])_{\partial T} \\
& - (\Delta v_\mathcal{T}, [\![\underline{\partial_{\boldsymbol{\nu}} u_h}]\!])_{\partial T} + \eta_1 h^{-1}([\![\underline{\partial_{\boldsymbol{\nu}} u_h}]\!], [\![\underline{\partial_{\boldsymbol{\nu}} v_h}]\!])_{\partial T} \\
a_h^{D^2}(u_h, v_h) := \sum_{T\in\mathcal{T}_h} & (\boldsymbol{n}^\top(D^2 u_\mathcal{T})\boldsymbol{n}, \Delta v_\mathcal{T})_T - (\boldsymbol{n}^\top(D^2 u_\mathcal{T})\boldsymbol{n}, [\![\underline{\partial_{\boldsymbol{\nu}} v_h}]\!])_{\partial T} \\
& - (\boldsymbol{n}^\top(D^2 v_\mathcal{T})\boldsymbol{n}, [\![\underline{\partial_{\boldsymbol{\nu}} u_h}]\!])_{\partial T} + \eta_2 h^{-1}([\![\underline{\partial_{\boldsymbol{\nu}} u_h}]\!], [\![\underline{\partial_{\boldsymbol{\nu}} v_h}]\!])_{\partial T} \\
c_h^{\Gamma_R}(u_h, v_h) := & -\alpha(\Delta u_\mathcal{T}, \mu_F)_{\mathcal{F}_h^R} + \alpha i\theta(\Delta u_\mathcal{T}, v_\mathcal{T})_{\mathcal{F}_h^R} + \beta i\theta(\boldsymbol{n}^\top(D^2 u_\mathcal{T})\boldsymbol{n}, v_\mathcal{T})_{\mathcal{F}_h^R} \\
& - \beta(\boldsymbol{n}^\top(D^2 u_\mathcal{T})\boldsymbol{n}, \mu_F)_{\mathcal{F}_h^R} - i\theta(u_\mathcal{T}, v_\mathcal{T})_{\mathcal{F}_h^R},
\end{aligned}$$

Weakly coercive operators

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- Suppose that $a(\cdot,\cdot) : V \times V \to \mathbb{C}$ is such that $a(\cdot,\cdot) = b(\cdot,\cdot) + c(\cdot,\cdot)$ where
  - $a(\cdot,\cdot)$ is coercive: $\Re\{a(u,u)\} \geq \gamma \|u\|_V^2$ for all $u \in V$;
  - $c(\cdot,\cdot)$ is compact, i.e. the associated operator $K \in \mathcal{L}(X)$ is compact.

  → special case: Gårding's inequality
- Then the operator $A \in \mathcal{L}(X)$ assoc. w. $a(\cdot,\cdot)$ is Fredholm with index zero, i.e.

  well-posedness $\Leftrightarrow$ injectivity (assumed from now on)
- Conforming Galerkin approximation: find $u_h \in V_h \subset V$ s.t.

$$a(u_h, v_h) = f(v_h) \qquad \forall v_h \in V_h.$$

Theorem [Demkowicz 1994, Sauter & Schwab 2011, Spence 2014, ...]

There exists $h_0 > 0$ s.t. $a(\cdot,\cdot)$ is inf-sup stable on $V_h \times V_h$ for $h \leq h_0$ and

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V.$$

Non-conforming setting

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- Abstract non-conforming setting: Find $u_h \in V_h \not\subset V$ s.t.

$$a_h(u_h, v_h) = f(v_h) \in V_h \qquad \forall v_h \in V_h.$$

- Additional assumptions:
  - regularity: $u \in V_*$ s.t. $\|u\|_{V_h}$, $a_h(u, v_h)$ make sense
  - continuity: $\exists \|\cdot\|_{V_h,*}$ s.t. $\|\cdot\|_{V_h} \leq \|\cdot\|_{V_h,*}$ and $a_h(u, v_h) \leq C\|u\|_{V_h,*}\|v_h\|_{V_h}$ $\forall u \in V_* + V_h, v_h \in V_h$
  - consistency: $a_h(u - u_h, v_h) = 0 \; \forall v_h \in V_h$

Theorem (Non-conforming approx.)

Let $a = b + c$ be injective, b coercive, c compact, and $a_h = b_h + c_h$ be s.t.

- $a_h(\cdot, \cdot)$ consistent with $a(\cdot, \cdot)$, $b_h(\cdot, \cdot)$ consistent with $b(\cdot, \cdot)$;
- $b_h(\cdot, \cdot)$ is uniformly coercive;
- $c_h(\cdot, \cdot)$ collectively compact (might be possible to remove);

Then, there exists $h_0 > 0$ s.t. $a_h(\cdot, \cdot)$ is inf-sup stable on $V_h \times V_h$ for $h \leq h_0$.

Continuous level: Weak coercivity

- For $\tilde{C} > \alpha$, we set

$$b(u,v) := \alpha(\Delta u, \Delta v)_\Omega + \beta(\boldsymbol{n}^\top(D^2u)\boldsymbol{n}, \Delta v)_\Omega + (\nabla u, \nabla v)_\Omega + \tilde{C}(u,v)_\Omega,$$
$$c(u,v) := -(k^2 - \tilde{C})(u,v)_\Omega + c^{\Gamma_R}(u,v).$$

- Then $a(u,v) = b(u,v) + c(u,v)$ and
  1. $b(\cdot,\cdot)$ is coercive,
  2. $c(\cdot,\cdot)$ is compact,
  3. $a(\cdot,\cdot)$ is injective for non-resonant k^2.
- 2. uses $H^2(\Omega) \hookrightarrow L^2(\Omega)$ compact & compactness of tr on Lipschitz doms
- For 1. we show that $b(u,u) \simeq \|u\|_{H^2}^2$ for β small enough, crucial estimate:

$$\|D^2u\|_\Omega^2 \leq C_{\Delta,D^2}\left(\|\Delta u\|_\Omega^2 + \|u\|_\Omega^2\right) \quad \forall u \in H^2(\Omega).$$

elliptic reg. of Δ + Peetre–Tartar theorem

Crucial tool: Enrichment operator

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- There exists $E_h : V_h \subset H^1(\Omega) \to H^2(\Omega)$ such that

$$\|D^2(v_\mathcal{T} - E_h v_\mathcal{T})\|_{\mathcal{T}_h} \leq C h^{-1/2} \|[\![\partial_{\boldsymbol{\nu}} v_\mathcal{T}]\!]_{\mathrm{DG}}\|_{L^2(\mathcal{F}_h^i)}. \tag{1}$$

Lemma

Let $h \leq 1$. For all $u_h \in V_h = \mathbb{P}^p(\mathcal{T}_h) \cap H^1(\Omega) \times \mathbb{P}^{p-1}(\mathcal{F}_h)$, we have that

$$\|D^2 u_\mathcal{T}\|_{\mathcal{T}_h}^2 \leq C_{\Delta, D^2, h} \left(\|\Delta u_\mathcal{T}\|_{\mathcal{T}_h}^2 + \|u_\mathcal{T}\|_\Omega^2 + \|h^{-1/2} [\![\underline{\partial_{\boldsymbol{\nu}} u_h}]\!]\|_{\partial\mathcal{T}_h}^2 \right).$$

- Introduce discrete norms $\|u_h\| \simeq \|u_h\|_*$

$$\|u_h\|^2 := \|u_h\|_{H^2(\mathcal{T}_h)}^2 + \|h^{-1/2} [\![\underline{\partial_{\boldsymbol{\nu}} u_h}]\!]\|_{\partial\mathcal{T}_h}^2 \qquad \text{(stability)}$$

$$\|u_h\|_*^2 := \|u_h\|^2 + \|h^{1/2} \Delta u_\mathcal{T}\|_{\partial\mathcal{T}_h}^2 + \|h^{1/2} \boldsymbol{n}^\top (D^2 u_\mathcal{T}) \boldsymbol{n}\|_{\partial\mathcal{T}_h}^2 \qquad \text{(continuity)}$$

Brenner, Sung, *Virtual enrichment operators.* Calcolo, 2019.

Stability

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- We split $a_h(\cdot,\cdot) = b_h(\cdot,\cdot) + c_h(\cdot,\cdot)$ with $\tilde{C} > \alpha$

$$b_h(u_h, v_h) := \alpha a_h^{\Delta}(u_h, v_h) + \beta a_h^{D^2}(u_h, v_h) + (\nabla u_\mathcal{T}, \nabla v_\mathcal{T})_\Omega + \tilde{C}(u_\mathcal{T}, v_\mathcal{T})_\Omega,$$
$$c_h(u_h, v_h) := -(k^2 - \tilde{C})(u_\mathcal{T}, v_\mathcal{T})_\Omega + c_h^{\Gamma_R}(u_h, v_h)$$

- Assume $u \in V_* := \{H^{2+\epsilon}(\Omega) : u|_{\Gamma_D} = 0, \epsilon > 0 \text{ s.t. } \Delta u, \boldsymbol{n}^\top (D^2 u)\boldsymbol{n} \in H^1(\Omega)\}$
  ➔ $u \mapsto (u, \mathrm{tr}(\partial_\nu u))$ s.t. $[\![\partial_{\boldsymbol{\nu}} \underline{u}]\!] = 0$, then $a_h(\cdot,\cdot)$ & $b_h(\cdot,\cdot)$ are consistent

For β small enough, $\eta_1, \eta_2 > 0$ large enough, we obtain uniform coercivity:

$$b_h(u_h, u_h) \geq \left(\frac{\alpha}{2} C_{\Delta, D^2, h}^{-1} - \frac{\beta\sqrt{d}}{2} \right) \|D^2 u_\mathcal{T}\|_{\mathcal{T}_h}^2 + \|\nabla u_\mathcal{T}\|_\Omega^2 + \left(\tilde{C} - \alpha/2\right) \|u_\mathcal{T}\|_\Omega^2$$
$$+ \left(\alpha\left(\eta_1 - C_{\mathrm{tr}}^2/2\right) + \beta\left(\eta_2 - C_{\mathrm{tr}}^2/2\right) - \alpha/2\right) \|h^{-1/2} [\![\partial_{\boldsymbol{\nu}} \underline{u_h}]\!]\|_{\partial \mathcal{T}_h}^2$$
$$\geq \gamma \|\!| u_h \|\!|^2$$

Implementational aspects

- Optimization: $V_{\mathcal{F}_h} = \mathbb{P}^{p-2}(\mathcal{F}_h)$ and Lehrenfeld–Schöberl stabilization

$$\eta_i h^{-1}(\llbracket \partial_\nu u_h \rrbracket, \llbracket \partial_\nu v_h \rrbracket)_{\partial\mathcal{T}_h} \quad \rightsquigarrow \quad \eta_i h^{-1}(\Pi_F^{p-2}\llbracket \partial_\nu u_h \rrbracket, \Pi_F^{p-2}\llbracket \partial_\nu v_h \rrbracket)_{\partial\mathcal{T}_h}$$

- Computational efficiency (with LS-stab., static cond. for hybrid version)

	C^0-hybrid IP		C^0-IP	
order	dofs	nzes	dofs	nzes
3	20 199	327 195	8707	346 507
5	41 287	1 104 475	23 991	1 990 431
7	69 959	2 338 651	46 859	6 648 251
9	106 215	4 029 723	77 311	16 740 991

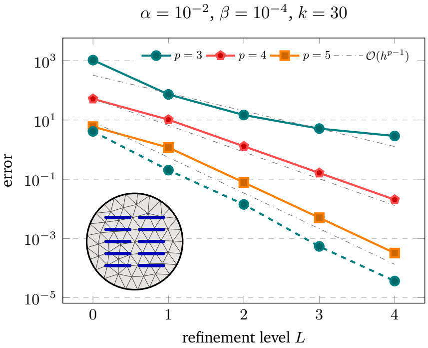
- for biharmonic problem: Hybridization of C^0-IP yields better condition number
- Helmholtz like character → preconditioning & iterative solution more tricky

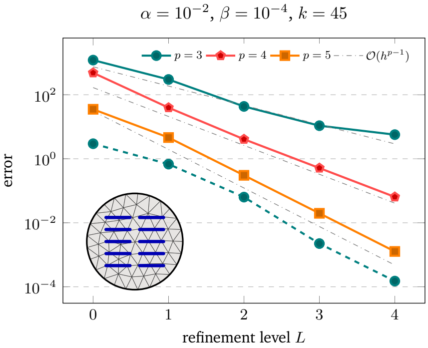
Lehrenfeld, Schöberl. *High order exactly div.-free HDG methods [...]* CMAME, 2016.

Dong, Ern, *C^0-HHO for biharmonic problems*, IMAJNA, 2024.

Numerical example: Convergence

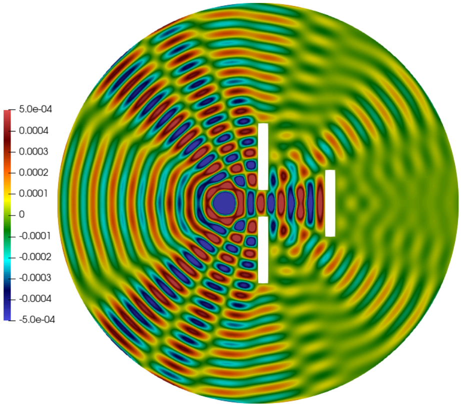
- We expect: $\|u - u_h\|_{H^2(\mathcal{T}_h)} \lesssim \inf_{v_h \in V_h} \|\!|u - v_h|\!\|_* \lesssim h^{p-1} \|u\|_{H^p(\Omega)}$
- plane wave ansatz: $u_{\text{ex}}(\boldsymbol{x}) := \exp(i\boldsymbol{d} \cdot \boldsymbol{x})$ for specific wave-vector $\boldsymbol{d} \in \mathbb{C}^d$

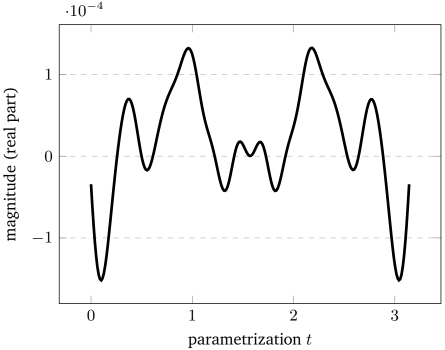




Numerical example

- scattering by sound-soft obstacles, magnitude measured at $(\sin(t), \cos(t))$





- nematic field: $\boldsymbol{n} = (0, 0)$

Numerical example

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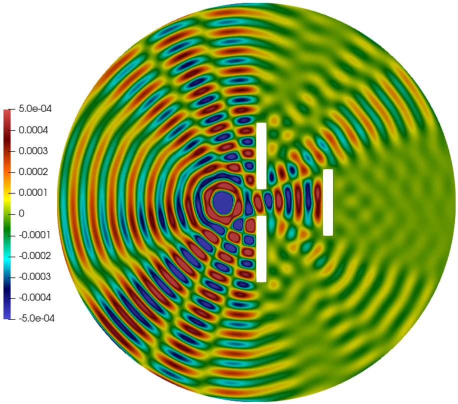
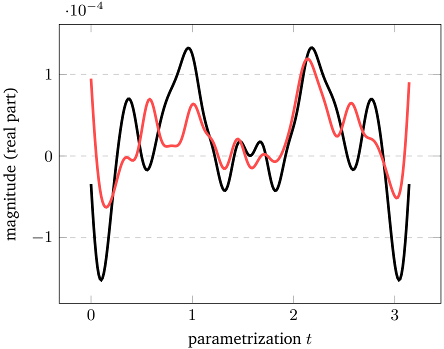
- scattering by sound-soft obstacles, magnitude measured at $(\sin(t), \cos(t))$

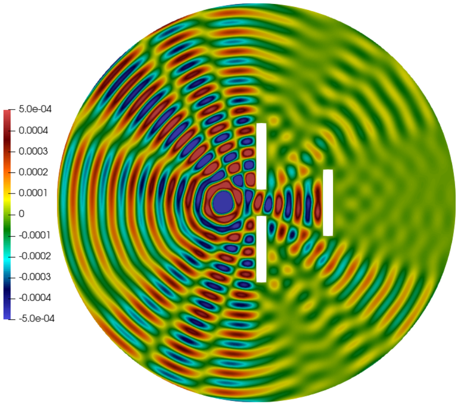


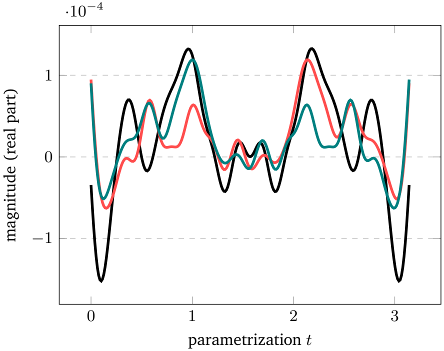


- nematic field: $\boldsymbol{n} = (1, -1)$

Numerical example

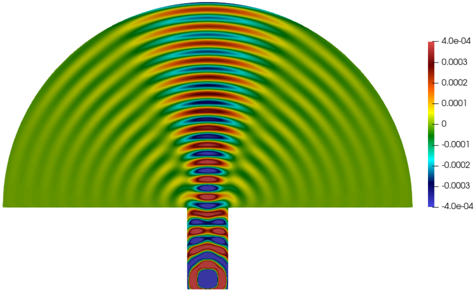
- scattering by sound-soft obstacles, magnitude measured at $(\sin(t), \cos(t))$



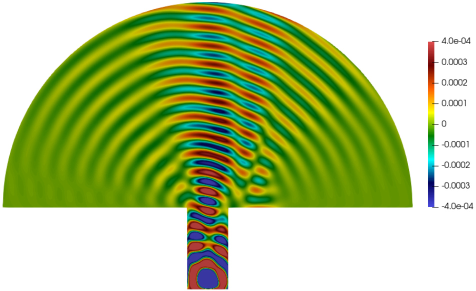


- nematic field: $\boldsymbol{n} = (1, 1)$

Numerical example: Wave guide

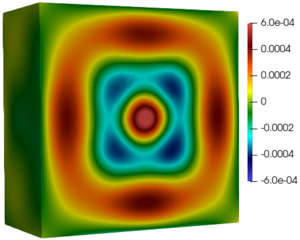


$\boldsymbol{n} = (0, 0)$

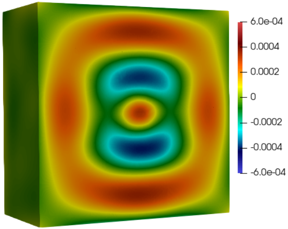


$\boldsymbol{n} = (1, -1)$

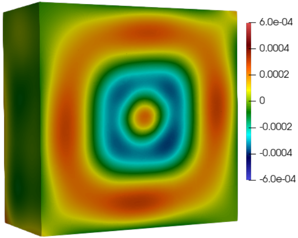
Numerical example: 3D



$\boldsymbol{n} = (0, 0, 0)$



$\boldsymbol{n} = (0, 1, 0)$



$\boldsymbol{n} = (1, 1, 1)$

Conclusion

- A conforming discretization of the NHK equation requires C^1-conforming elements, but they can be difficult to handle
- C^0-hybrid interior penalty is non-conforming alternative that offers more flexibility, in particular:
  - stability & quasi-optimality for $p \geq 2$ in 2D and 3D,
  - hybridization reduces computational cost

Thank you for your attention!