Stable (hybrid) discontinuous Galerkin discretizations for Galbrun's equation

Martin Halla[1], Christoph Lehrenfeld[2], Tim van Beeck[2]

[1]RICAM, Austrian Academy of Sciences, Linz
[2]Institute for Numerical and Applied Mathematics, University of Göttingen

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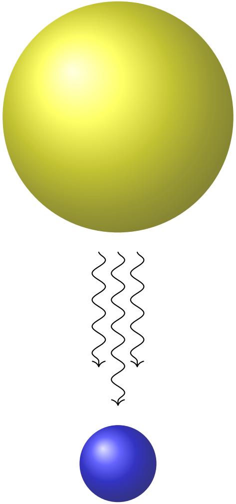
Galbrun's equation

Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$ s.t.

$$
\begin{aligned}
&-\nabla(c_s \rho \operatorname{div} \boldsymbol{u}) + (\operatorname{div} \boldsymbol{u}) \nabla p - \nabla(\nabla p \cdot \boldsymbol{u}) \\
&-\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} \\
&+ (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\Phi))\boldsymbol{u} - \gamma\rho i\omega \boldsymbol{u} = \boldsymbol{f} \text{ in } \mathcal{O}, \\
&\qquad\qquad\qquad\qquad \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \text{ on } \partial\mathcal{O}.
\end{aligned}
$$

c_s: sound speed, ρ: density, $\boldsymbol{b}$: background flow, γ: damping factor, ω: frequency, Ω: rotation, p: pressure, Φ: grav. potential



Challenges

- nonstandard differential operator $\partial_{\boldsymbol{b}} := \sum_{l=1}^{3} \boldsymbol{b}_l \partial_{x_l}$
- indefinite problem
- strongly varying coefficients
- high computational cost

Outline

T-coercivity

Theorem ([1])

Let X be Hilbert and $a(\cdot,\cdot) : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ be a bounded sesquilinear form. The problem

$$\textit{find } u \in \mathbb{X} \textit{ s.t. } a(u,v) = f(v) \quad \forall v \in \mathbb{X}$$

is well-posed iff $\exists T : \mathbb{X} \to \mathbb{X}$ bijective s.t. $a(T\cdot,\cdot)$ is coercive, i.e.

$$\Re\{a(Tu,u)\} \geq \alpha \|u\|_X^2 \quad \forall u,v \in \mathbb{X}.$$

Note: For Hilbert spaces, T-coercivity is equivalent to the inf sup-condition.

- ➔ necessary & sufficient condition for well-posedness
- ➔ not automatically transferred to the discrete level

[1] see e.g., A.S. Bonnet-Ben Dhia, P. Ciarlet, C.M. Zwölf, "Time harmonic wave diffraction problems in materials with sign-shifting coefficients, 2010.

Let $A \in L(\mathbb{X})$ be the associated operator to $a(\cdot,\cdot)$, i.e.

$$\langle Au, v\rangle_{\mathbb{X}} := a(u,v) \quad \forall u, v \in \mathbb{X}.$$

Then A is T-coercive, if there exists $T \in L(\mathbb{X})$ bijective s.t. AT is coercive.

Definition (weak T-coercivity)

$A \in L(\mathbb{X})$ is weakly T-coercive, if $\exists T \in L(\mathbb{X})$ bijective and $K \in L(\mathbb{X})$ compact s.t. $AT + K$ is coercive.

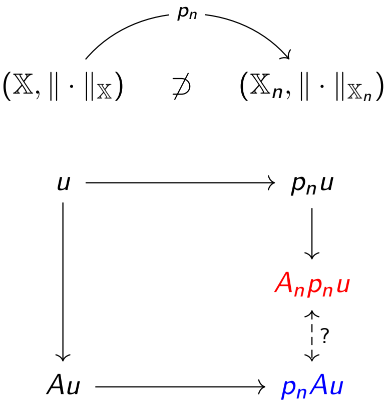
- weakly T-coercive $\Rightarrow$ Fredholm with index zero
- weakly T-coercive + injective $\Rightarrow$ bijective

Approximation of weakly T-coercive operators I/II

Definition (Discrete approximation scheme[2])

$(\mathbb{X}_n, p_n, A_n)$ is a DAS of $(\mathbb{X}, A)$ if

- $p_n \in L(\mathbb{X}, \mathbb{X}_n)$ s.t. $\lim_{n\to\infty} \|p_n u\|_{\mathbb{X}_n} = \|u\|_{\mathbb{X}}\ \forall u \in \mathbb{X}$,
- A_n *approximates* A, i.e. $\lim_{n\to\infty} \|(A_n p_n - p_n A)u\|_{\mathbb{X}_n} = 0\ \forall u \in \mathbb{X}$.



Example

A conforming Galerkin scheme, $\mathbb{X}_n \subset \mathbb{X}$, $p_n = \Pi^{L^2}_{\mathbb{X}_n}$, $A_n = (\Pi^{L^2}_{\mathbb{X}_n} A_n)|_{\mathbb{X}_n}$ is always a discrete approximation scheme.

[2] goes back to (among others) Vainikko (1976), Stummel (1970,1971), Karma (1996).

Approximation of weakly T-coercive operators II/II

Assumptions

- $(\mathbb{X}_n, p_n, A_n)$ is a DAS of $(\mathbb{X}, A)$
- $A = B + K$, B bijective, K compact
- A injective

$$\begin{array}{ccccc} AT & = & B & + & K \\ \uparrow & & \uparrow & & \\ A_n T_n & = & B_n & + & K_n \end{array}$$

Theorem (Weak T-compatibility[3])

If there exists sequences $(A_n)_{n\in\mathbb{N}}, (T_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}}$ *such that* $(T_n)_{n\in\mathbb{N}}$ *and* $(B_n)_{n\in\mathbb{N}}$ *are uniformly stable,* $(K_n)_{n\in\mathbb{N}}$ *compact and*

$$\lim_{n\to\infty} \|(T_n p_n - p_n T)u\|_{\mathbb{X}_n} = 0, \quad \lim_{n\to\infty} \|(B_n p_n - p_n B)u\|_{\mathbb{X}_n} = 0, \quad A_n T_n = B_n + K_n,$$

then $(A_n)_{n\in\mathbb{N}}$ *is stable, i.e.* A_n^{-1} *exists,* $\|A_n^{-1}\|_{L(\mathbb{X}_n)} \leq C$ *for all* $n > n_0$.

[3] M. Halla, C. Lehrenfeld, P. Stocker, "A new T-compatibility condition and its application to the discretization of the damped time-harmonic Galbrun's equation", 2022.

Back to Galbrun's equation

Define

$$\mathbb{X} := \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \operatorname{div} \boldsymbol{u} \in L^2(\mathcal{O}), \partial_{\boldsymbol{b}} \boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}), \boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\mathcal{O}\},$$

$$\langle \boldsymbol{u}, \boldsymbol{u}' \rangle_{\mathbb{X}} := \langle \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle_{L^2} + \langle \boldsymbol{u}, \boldsymbol{u}' \rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}} \boldsymbol{u}, \partial_{\boldsymbol{b}} \boldsymbol{u}' \rangle_{\boldsymbol{L}^2}.$$

Weak formulation

Find $\boldsymbol{u} \in \mathbb{X}$ s.t. $a(\boldsymbol{u}, \boldsymbol{u}') = \langle \boldsymbol{f}, \boldsymbol{u}' \rangle \; \forall \boldsymbol{u}' \in \mathbb{X}$, where

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{u}') :=& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}' \rangle \\ &+ \langle \operatorname{div} \boldsymbol{u}, \nabla p \cdot \boldsymbol{u}' \rangle - \langle \nabla p \cdot \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}, \boldsymbol{u}' \rangle \\ &- i\omega \langle \rho\gamma \boldsymbol{u}, \boldsymbol{u}' \rangle. \end{aligned}$$

→ Well-posed[4] if $\|c_s^{-1} \boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \lesssim 1$.

[4] M. Halla, T. Hohage, "On the well-posedness of the damped time-harmonic Galbrun equation and the equations of solar and stellar oscillation", 2021.

Dealing with the indefiniteness

"Simplified" Galbrun (p & ϕ constant, $\Omega = 0$):

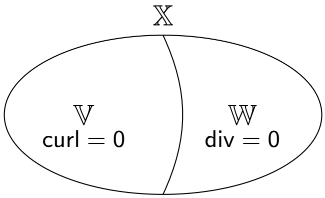
$$a(\boldsymbol{u}, \boldsymbol{u}') = \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}})\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}})\boldsymbol{u}' \rangle - i\omega \langle \rho\gamma \boldsymbol{u}, \boldsymbol{u}' \rangle = \langle \boldsymbol{f}, \boldsymbol{u}' \rangle.$$

Problem: if $\boldsymbol{u} \in \ker(\operatorname{div})$, then

$$\Re\{a(\boldsymbol{u}, \boldsymbol{u})\} = -\|\rho^{1/2}(\omega + i\partial_{\boldsymbol{b}})\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 \not\gtrsim \|\boldsymbol{u}\|_{\mathbb{X}}^2$$

Idea: Flip the problematic sign! For $\boldsymbol{u} \in \mathbb{X}$, find $v \in H_*^2$ .s.t.

$$\operatorname{div} \nabla v = \operatorname{div} \boldsymbol{u} \text{ in } \mathcal{O},$$
$$\boldsymbol{\nu} \cdot \nabla v = 0 \text{ on } \partial\mathcal{O}$$



and set $\boldsymbol{v} = \nabla v$, $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, $T\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{w}$.
If $\boldsymbol{u} \in \ker(\operatorname{div})$, i.e. $\boldsymbol{u} = \boldsymbol{w}$, then

$$\Re\{a(T\boldsymbol{u}, \boldsymbol{u})\} = \|\rho^{1/2}(\omega + i\partial_{\boldsymbol{b}})\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 \gtrsim \|\boldsymbol{u}\|_{\mathbb{X}}^2$$

Full equation

With $\boldsymbol{q} := c_s^{-2}\rho^{-1}\nabla p$, we write

$$a(\boldsymbol{u},\boldsymbol{u}') := \langle c_s^2\rho(\operatorname{div}+\boldsymbol{q}\cdot)\boldsymbol{u},(\operatorname{div}+\boldsymbol{q}\cdot)\boldsymbol{u}'\rangle - \langle\rho(\omega+i\partial_{\boldsymbol{b}}+i\Omega\times)\boldsymbol{u},(\omega+i\partial_{\boldsymbol{b}}+i\Omega\times)\boldsymbol{u}'\rangle$$
$$+\langle(\operatorname{Hess}(p)-\operatorname{Hess}(\phi)-c_s^2\rho\boldsymbol{q}\otimes\boldsymbol{q})\boldsymbol{u},\boldsymbol{u}'\rangle - i\omega\langle\rho\gamma\boldsymbol{u},\boldsymbol{u}'\rangle.$$

Decomposition of $\mathbb{X}$

For $\boldsymbol{u}\in\mathbb{X}$, find $v\in H^2_*$ s.t.

$$(\operatorname{div}+P_{L^2_0}\boldsymbol{q}\cdot+M)\nabla v = (\operatorname{div}+P_{L^2_0}\boldsymbol{q}\cdot+M)\boldsymbol{u} \text{ in } \mathcal{O},$$
$$\boldsymbol{\nu}\cdot\nabla v = 0 \text{ on } \partial\mathcal{O},$$

where $P_{L^2_0}$ is the orthogonal projection onto L^2_0 and M is a suitable operator of finite rank. Set $\boldsymbol{v} := \nabla v$, $\boldsymbol{w} := \boldsymbol{u}-\boldsymbol{v}$, $T\boldsymbol{u} := \boldsymbol{v}-\boldsymbol{w}$. Then A is weakly T-coercive.

Discrete stability

Goal: transfer the decomposition of $\mathbb{X}$ to the discrete level
Stability closely related to stable pairs for Stokes

- H^1-conforming discretization[5]
  - might need special meshes (barycentric refinement)
  - and / or k large enough
- $H(\mathrm{div})$-conforming DG[6]
- fully nonconforming DG[7]

→ now: hybrid DG

$$\mathbb{X}_h = \mathbb{V}_h \oplus \mathbb{W}_h$$

$$\boldsymbol{u}_h = \boldsymbol{v}_h + \boldsymbol{w}_h$$

[5] M. Halla, C. Lehrenfeld, P. Stocker, "A new T-compatibility condition and its application to the discretization of the damped time-harmonic Galbrun's equation", 2022.

[6] M. Halla, "Convergence analysis of nonconform $H(\mathrm{div})$-finite elements for the damped time-harmonic Galbrun's equation", 2023.

[7] TvB, "On stable discontinuous Galerkin discretizations for Galbrun's equation", 2023.

(Hybrid) $H(\mathrm{div})$-conforming DG

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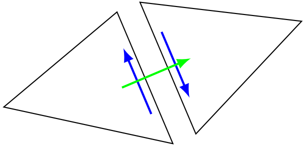
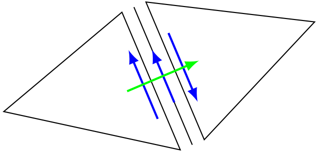
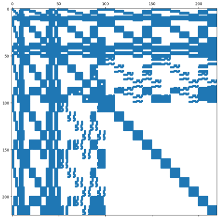
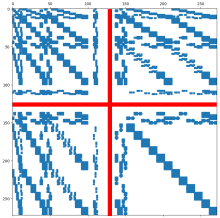
	$H(\mathrm{div})$**-conf. DG**	$H(\mathrm{div})$**-conf. HDG**
dofs		
spaces	$\mathbb{BDM}^k(\mathcal{T}_n)$	$\mathbb{BDM}^k(\mathcal{T}_n) \times [\Pi^{\mathrm{tang}}(\mathbb{P}^k(\mathcal{F}_n))]^d$
sparsity	 $\in \mathbb{R}^{220\times 220}$	 $\in \mathbb{R}^{272\times 272}$

Discretization

Let $\mathcal{T}_n$ be a shape regular, quasi-uniform triangulation of $\mathcal{O}$ and $\mathcal{F}_n$ the set of faces. We set

$$\mathbb{X}_n := \mathbb{BDM}^k(\mathcal{T}_n) \times [\mathbb{P}^k(\mathcal{F}_n)]^d$$
$$\langle \boldsymbol{u}_n, \boldsymbol{u}'_n \rangle_{\mathbb{X}_n} := \langle \operatorname{div} \boldsymbol{u}_\mathcal{T}, \operatorname{div} \boldsymbol{u}'_\mathcal{T} \rangle_{L^2} + \langle \boldsymbol{u}_\mathcal{T}, \boldsymbol{u}'_\mathcal{T} \rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}_n, \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}'_n \rangle_{\boldsymbol{L}^2},$$

where $\boldsymbol{D}^n_{\boldsymbol{b}}$ is a discrete version of $\partial_{\boldsymbol{b}}$ defined lateron.

For $\boldsymbol{u} \in \mathbb{X}$, we define $p_n \boldsymbol{u} \in \mathbb{X}_n$ as the solution to

$$\langle p_n \boldsymbol{u}, \boldsymbol{u}'_n \rangle_{\mathbb{X}_n} = \langle \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}'_\mathcal{T} \rangle + \langle \boldsymbol{u}, \boldsymbol{u}'_\mathcal{T} \rangle + \langle \partial_{\boldsymbol{b}} \boldsymbol{u}, \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}'_n \rangle \quad \forall \boldsymbol{u}'_n \in \mathbb{X}_n.$$

Lemma

For all $\boldsymbol{u} \in \mathbb{X}$, we have $\lim_{n\to\infty} \|p_n \boldsymbol{u}\|_{\mathbb{X}_n} = \|\boldsymbol{u}\|_{\mathbb{X}}$.

Stabilization

For $\boldsymbol{u}_n = (\boldsymbol{u}_\tau, \boldsymbol{u}_F) \in \mathbb{X}$ and $\tau \in \mathcal{T}_n$, set

$$[\![\underline{\boldsymbol{u}_n}]\!]_{\boldsymbol{b}} := (\boldsymbol{b} \cdot \nu)\boldsymbol{u}_\tau - (\boldsymbol{b} \cdot \nu)\boldsymbol{u}_F$$

Goal: stabilization without penalty parameter depending on flow $\boldsymbol{b}$

➔ standard SIP method leads to further Mach number restrictions

Definition (Lifting operator)

For $\boldsymbol{u}_n \in \mathbb{X}_n$, let $\boldsymbol{R}^l \boldsymbol{u}_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ be the solution to

$$\langle \boldsymbol{R}^l \boldsymbol{u}_n, \psi_n \rangle = - \sum_{\tau \in \mathcal{T}_n} \langle [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \psi_n \rangle_{\partial\tau} \quad \forall \psi_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d.$$

Then, we define for all $\tau \in \mathcal{T}_n$

$$(\boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}_n)|_\tau := \partial_{\boldsymbol{b}}(\boldsymbol{u}_n|_\tau) + \boldsymbol{R}^l \boldsymbol{u}_n.$$

Discrete problem

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Find $\boldsymbol{u}_n = (\boldsymbol{u}_\mathcal{T}, \boldsymbol{u}_F) \in \mathbb{X}_n$ s.t. $a_n(\boldsymbol{u}_n, \boldsymbol{u}'_n) = \langle \boldsymbol{f}, \boldsymbol{u}'_n \rangle$ for all $\boldsymbol{u}'_n \in \mathbb{X}_n$ with

$$\begin{aligned}
a_n(\boldsymbol{u}_n, \boldsymbol{u}'_n) :=& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_\mathcal{T}, \operatorname{div} \boldsymbol{u}'_\mathcal{T} \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}'_n \rangle \\
&+ \langle \operatorname{div} \boldsymbol{u}_\mathcal{T}, \nabla p \cdot \boldsymbol{u}'_\mathcal{T} \rangle + \langle \nabla p \cdot \boldsymbol{u}_\mathcal{T}, \operatorname{div} \boldsymbol{u}'_\mathcal{T} \rangle \\
&+ \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_\mathcal{T}, \boldsymbol{u}'_\mathcal{T} \rangle - i\omega \langle \gamma \rho \boldsymbol{u}_\mathcal{T}, \boldsymbol{u}'_\mathcal{T} \rangle \\
&- \langle c_s^2 \rho \boldsymbol{u}_\mathcal{T}, [\![\underline{\boldsymbol{u}'_n}]\!]_\nu \rangle_{\partial\mathcal{T}_n} - \langle c_s^2 \rho [\![\underline{\boldsymbol{u}_n}]\!]_\nu, \boldsymbol{u}'_\mathcal{T} \rangle_{\partial\mathcal{T}_n} + \langle c_s^2 \rho \frac{\alpha}{h} [\![\underline{\boldsymbol{u}_n}]\!]_\nu, [\![\underline{\boldsymbol{u}'_n}]\!]_\nu \rangle_{\partial\mathcal{T}_n}
\end{aligned}$$

Lemma

We have that $\lim_{n\to\infty} \|(A_n p_n - p_n A)\boldsymbol{u}\|_{\mathbb{X}_n} = 0$ *for all* $\boldsymbol{u} \in \mathbb{X}$.

Proof.

Main idea: For a bounded sequence $(u_n)_{n\in\mathbb{N}}$ $\exists\, \boldsymbol{u} \in \mathbb{X}$ s.t. $\boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{u}_n \overset{L^2}{\rightharpoonup} \partial_{\boldsymbol{b}} \boldsymbol{u}$. □

Discrete decomposition

Take $\boldsymbol{u}_n = (\boldsymbol{u}_\mathcal{T}, \boldsymbol{u}_F) \in \mathbb{X}_n$.
Decomposition of volume part: Find $v \in H^2_*$ s.t.

$$(\operatorname{div} + P_{L^2_0}\boldsymbol{q} \cdot + M)\nabla\tilde{v} = (\operatorname{div} + \pi^l_n \boldsymbol{q} \cdot + M)\boldsymbol{u}_\mathcal{T} \text{ in } \mathcal{O},$$
$$\boldsymbol{\nu} \cdot \nabla\tilde{v} = 0 \text{ on } \partial\mathcal{O},$$

- → well-posed as $\boldsymbol{u}_\mathcal{T} \in \mathbb{BDM}^k \subset H(\operatorname{div})$ $(\boldsymbol{u}_\mathcal{T} \in \mathbb{P}^k(\mathcal{T}_n) \to \boldsymbol{u}_\mathcal{T} - \mathcal{I}_{[\![\cdot]\!]_\nu}(\boldsymbol{u}_\mathcal{T}) \in H(\operatorname{div}).)$
- → Set $\boldsymbol{v}_\mathcal{T} := \pi^d_n \nabla\tilde{v}$ and $\boldsymbol{w}_\mathcal{T} := \boldsymbol{u}_\mathcal{T} - \boldsymbol{v}_\mathcal{T}$

Decomposition of facet part: Set $\boldsymbol{v}_F := \boldsymbol{u}_F - \operatorname{tr}_{\mathcal{F}_n} \boldsymbol{w}_\mathcal{T}$ and $\boldsymbol{w}_F := \operatorname{tr}_{\mathcal{F}_n} \boldsymbol{w}_\mathcal{T}$.
Definition of T_n: Set $T_n \boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n$.

Lemma

For all $\boldsymbol{u} \in \mathbb{X}$, *it holds that* $\lim_{n\to\infty} \|(T_n p_n - p_n T)\boldsymbol{u}\|_{\mathbb{X}_n} = 0$.

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Theorem

Assume that $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \lesssim 1$. *Then the remaining weak T-compatibility conditions are satisfied for the discrete problem, i.e.* $\exists (B_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}}$ *s.t.* $(B_n)_{n\in\mathbb{N}}$ *is uniformly stable,* $(K_n)_{n\in\mathbb{N}}$ *is compact,* $A_n T_n = B_n + K_n$ *and for all* $\boldsymbol{u} \in \mathbb{X}$, *we have*

$$\lim_{n\to\infty} \|(B_n p_n - p_n B)\boldsymbol{u}\|_{\mathbb{X}_n} = 0.$$

Corollary

There exists an index n_0 *s.t. the discrete problem has a unique solution for all* $n > n_0$. *If* $\boldsymbol{u} \in \mathbb{X} \cap \boldsymbol{H}^{2+s}$, $s > 0$, $\rho \in W^{1+s,\infty}$ *and* $\boldsymbol{b} \in \boldsymbol{W}^{1+s,\infty}$, *then*

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \le C \left(h^{\min(1+s,k)} + h^{\min(s,l)} \right) \|\boldsymbol{u}\|_{\boldsymbol{H}^{2+s}}.$$

Numerical experiments: Convergence

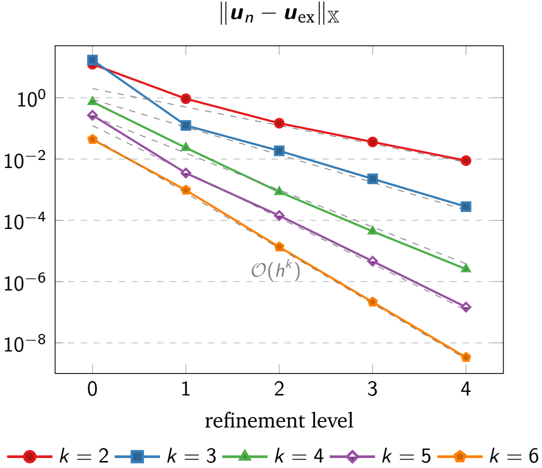
Set $\mathcal{O} = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$ and

$$\rho = \sqrt{10/\pi}\exp(-10(x^2 + y^2))$$
$$c_s^2 = 1.44 + 0.25\rho, \omega = 0.78 \times 2\pi,$$
$$\gamma = 0.1, \Omega = (0,0), p = 1.44\rho + 0.08\rho^2$$
$$\boldsymbol{b} = c_{\boldsymbol{b}} c_s \begin{pmatrix} -y \\ x \end{pmatrix}$$

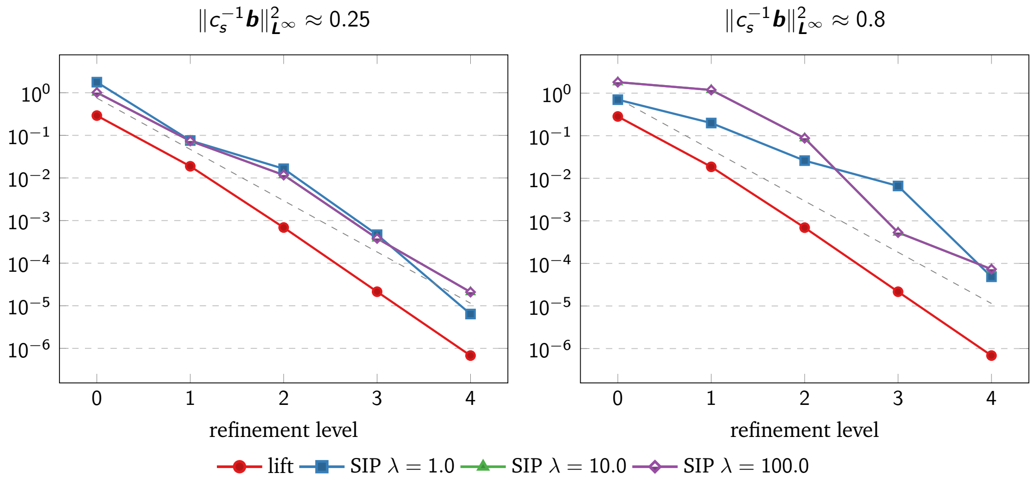
Calculate $\boldsymbol{f}$ s.t. the exact solution is

$$\boldsymbol{u}_{\text{ex}} = \frac{1}{\rho}\sin(r^2)\sin(r^2 - 1)\begin{pmatrix} +(1+i)g \\ -(1+i)g \end{pmatrix},$$

where $g = \sqrt{\alpha/\pi}\exp(-\alpha(x^2 + y^2))$, $\alpha = \log(10^9)$, is a Gaussian peak.

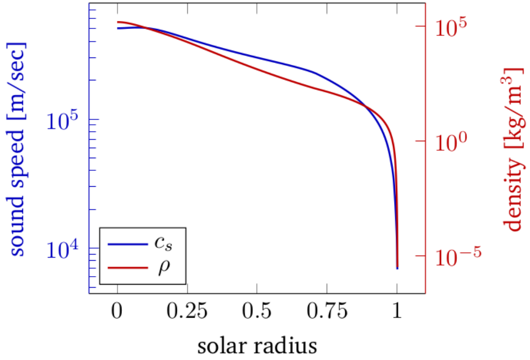


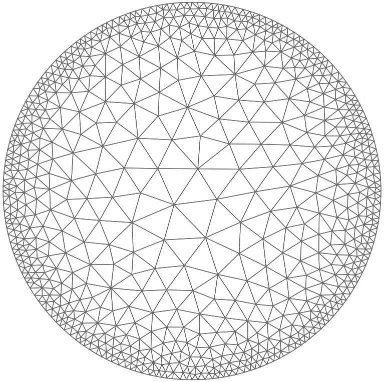
Numerical example: Lifting vs SIP



Numerical example: Sun parameters[8]

c_s, ρ, p from `modelS`, $\omega = 2\pi \cdot 0.003$, $\gamma = \omega/100$, $\boldsymbol{b} = 1/\rho \cdot 1.5/R \cdot (-y, x)^T$, $\boldsymbol{f} = (-i\omega + \partial_{\boldsymbol{b}})(g, 0)^T$





[8] adapted from: J. Chabassier and M. Duruflé. "Solving time-harmonic Galbrun's equation with an arbitrary flow. Application to Helioseismology", 2018.

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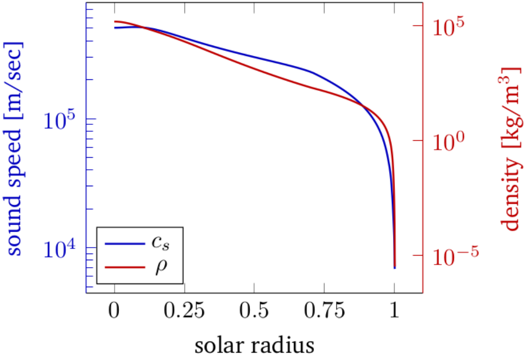

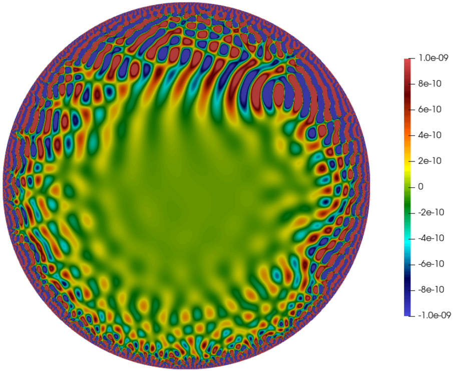

[8] adapted from: J. Chabassier and M. Duruflé. "Solving time-harmonic Galbrun's equation with an arbitrary flow. Application to Helioseismology", 2018.

Conclusions

Conclusions

- Discrete stability of Galbrun's equation related to stability of the Stokes complex
  - H^1-conforming discretization only stable for special meshes / higher k
  - ($H(\mathrm{div})$-conforming) (H)DG are stable
- Lifting operator gives stabilization without penalty parameter
- Numerical experiments confirm convergence rates & work with sun parameters

Outlok

- 3D simulation too expensive
- physically relevant boundary conditions

Thank you for your attention!