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CRC 1456 MATHEMATICS OF EXPERIMENT

Divergence-preserving unfitted FEM for the mixed Poisson problem

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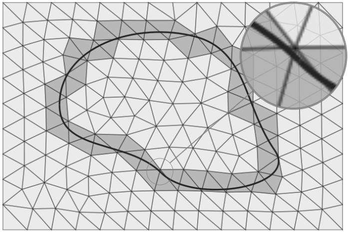
MS0304 Immersed boundary variational methods

Unfitted FEM

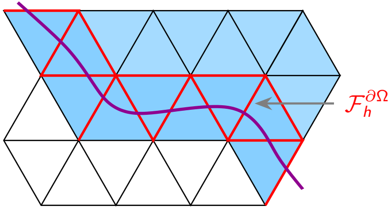
- → PDEs on embedded surfaces, moving domains, separate geometry description
- → Major challenge: Stability / Robustness w.r.t. arbitrarily small cuts

↓

Ghost penalty stabilization[1]



$$👻(u_h, v_h) := \sum_{F\in\mathcal{F}_h^{\partial\Omega}} \sum_{l=0}^{k} \frac{h_F^{2l+1}}{l!^2} \int_F [\![\partial_n^l u_h]\!][\![\partial_n^l v_h]\!]\, ds$$



- → Stability through: $\|u\|_{H^q(\Omega)} + |u|_{👻} \simeq \|u\|_{H^q(\Omega^{\mathcal{T}})}$

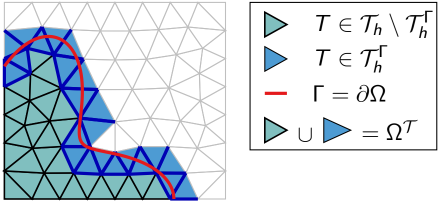
[1] E. Burman. Ghost penalty. C.R. Math., 348(21-22):1217–1220, November 2010.

Mixed Poisson problem

Model problem (Dirichlet case)

Find u, p with $p = p_D$ on $\partial\Omega$ s.t.

$$\begin{aligned} u - \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div} u &= -f \ \text{in } \Omega. \end{aligned}$$



Weak formulation - fitted mixed Poisson

Find $u_h \in \Sigma_h = \mathbb{RT}^k \subset H(\operatorname{div}, \Omega)$, $p_h \in Q_h = \mathbb{P}^k \subset L^2(\Omega)$ s.t.

$$\begin{aligned} (u_h, v_h)_\Omega + (\operatorname{div} v_h, p_h)_\Omega &= (v_h, p_D)_{\partial\Omega} \qquad \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_\Omega \qquad\qquad &= (-f, q_h)_\Omega \qquad \forall q_h \in Q_h. \end{aligned}$$

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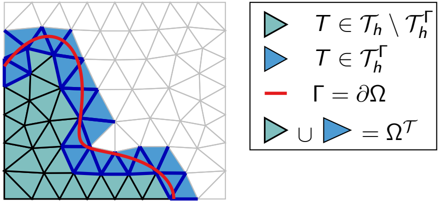

Weak formulation - naive unfitted mixed Poisson

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h) \subset H(\operatorname{div}, \Omega^\mathcal{T})$, $p_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h) \subset L^2(\Omega^\mathcal{T})$ s.t.

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➔ Not stable / robust! inf-sup constant degenerates for small cuts

Mixed Poisson problem

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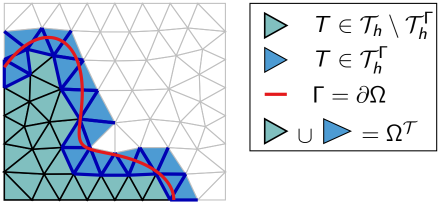

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$$\begin{aligned} (u_h, v_h)_\Omega + \text{👻} + (\operatorname{div} v_h, p_h)_\Omega + \text{👻} &= (v_h, p_D)_{\partial\Omega} \qquad \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_\Omega + \text{👻} \qquad\qquad &= (-f, q_h)_\Omega \qquad \forall q_h \in Q_h. \end{aligned}$$

➔ Stable, but Ghost penalty pollutes mass balance!

Literature

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Poisson / Darcy Problem:

Mass balance polluted, stability:

R. Puppi, *A cut finite element method for the Darcy problem.* arXiv: 2111.09922, 2021.

P. Cao, J. Chen, *An extended finite element method for coupled Darcy-Stokes problems.* IJNME, 2022

Mass balance hardly polluted, stability, divergence preserving:

T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A Divergence Preserving Cut Finite Element Method for Darcy Flow.* SIAM Journal of Sci. Comp., 2024.

Stokes Problem:

T. Frachon, E. Nilsson, Sara Zahedi, *Divergence-free cut finite element methods for Stokes flow.* arXiv:2304.14230, 2023.

Same idea as our approach:

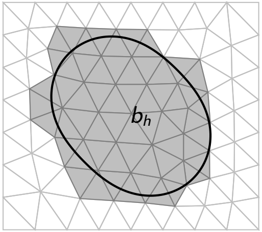
E. Burman, P. Hansbo, M. G. Larson, *Cut Finite Element Method for Divergence-Free Approximation of Incompressible Flow: A Lagrange Multiplier Approach.* SIAM Journal of Num. Anal., 2024.

Stable Unfitted Mixed Poisson

Stability of Saddle Point Problems[2]

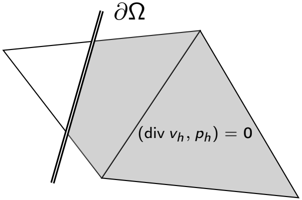
Set $a(u_h, v_h) := (u_h, v_h)_\Omega$ and $b(v_h, p_h) = (\operatorname{div} v_h, p_h)_\Omega$.

- ➔ coercivity of $a(\cdot,\cdot)$ on ker b:
  ✓ (since $a(u_h, u_h) = \|u_h\|_{H(\operatorname{div},\Omega)}$ on ker b).
- ➔ inf-sup stability of $b(\cdot,\cdot)$ on $\Sigma_h \times Q_h$:
  ✗ (constant degenerates for small cuts).



Let $b_h(v_h, p_h) := (\operatorname{div} v_h, p_h)_{\Omega^{\mathcal{T}}}$. Then:

$$\ker b = \ker b_h.$$



[2] cf. e.g., D. Boffi, F. Brezzi, M. Fortin, Mixed FEM and Applications, Springer, 2013.

Stable Unfitted Mixed Poisson

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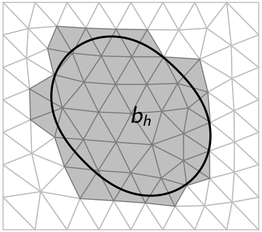
  ✓ (since $a(u_h, u_h) = \|u_h\|_{H(\operatorname{div},\Omega)}$ on ker b_h).
- → inf-sup stability of $b_h(\cdot,\cdot)$ on $\Sigma_h \times Q_h$:
  ✓ (constant independent of cut position)



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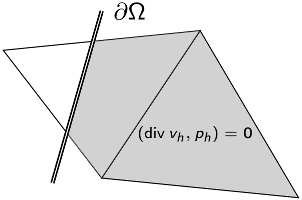


→ Replace b by b_h !

[2] cf. e.g., D. Boffi, F. Brezzi, M. Fortin, Mixed FEM and Applications, Springer, 2013.

Stable unfitted mixed Poisson

For $\gamma_{👻} \geq 0$, find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$ s.t.

$$\begin{aligned} (u_h, v_h)_\Omega + \gamma_{👻} 👻(u_h, v_h) + (\operatorname{div} v_h, \bar{p}_h)_{\Omega^{\mathcal{T}}} &= (v_h, p_D)_{\partial\Omega} && \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_{\Omega^{\mathcal{T}}} &= (-f_h, q_h)_{\Omega^{\mathcal{T}}} && \forall q_h \in Q_h. \end{aligned}$$

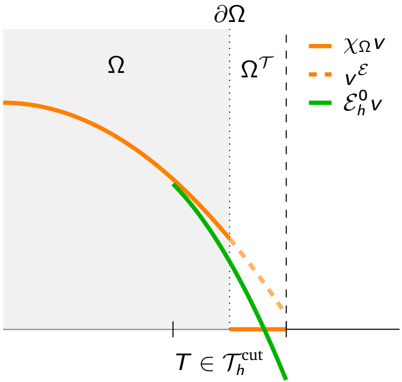
- f_h is a suitable discrete extension of f to $\Omega^{\mathcal{T}}$ (in the sense that it approximates $f^{\mathcal{E}}$ well)
- Inf-sup stable independent of cut position, consistent for u (up to $f_h \approx f$), consistent for p on interior elements
- But: inconsistent for p on cut elements

Interpretation of $\overline{p}_h$

Define $\mathcal{E}_h^0 : L^2(\Omega) \to Q_h$, $v \mapsto \Pi^Q(\chi_\Omega v)$ s.t.

$$(\mathcal{E}_h^0 q, r_h)_{\Omega^\mathcal{T}} = (q, r_h)_\Omega \qquad \forall r_h \in Q_h.$$

- ➔ $(\operatorname{div} v_h, q)_\Omega = (\operatorname{div} v_h, \mathcal{E}_h^0 q)_{\Omega^\mathcal{T}}$
- ➔ $\bar{p}_h \approx \mathcal{E}_h^0 p$
- ➔ $\bar{p}_h \approx p$ on $T \in \mathcal{T}_h^{\text{interior}}$
- ➔ $\bar{p}_h \not\approx p$ on $T \in \mathcal{T}_h^{\text{cut}}$



Error estimates

Norm on $H(\operatorname{div}, \Omega^{\mathcal{T}})$:

$$\|u\|_{\Sigma}^2 := \|\operatorname{div} u\|_{\Omega^{\mathcal{T}}}^2 + \|u\|_{\Omega_\gamma}^2, \qquad \|u\|_{\Omega_\gamma}^2 := \begin{cases} \|u\|_{\Omega^{\mathcal{T}}}^2 & \text{if } \gamma_{\text{👻}} > 0, \\ \|u\|_{\Omega}^2 & \text{if } \gamma_{\text{👻}} = 0. \end{cases}$$

$p^{\mathcal{E}}$ smooth Sobolev extension of p, $u^{\mathcal{E}} = \nabla p^{\mathcal{E}}$, $f^{\mathcal{E}} = -\operatorname{div} u^{\mathcal{E}}$.

Theorem (Error estimate for u_h)

For $u \in H^m(\Omega)$ with $m \in \{0, \dots, k+1\}$, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} + \|u^{\mathcal{E}} - u_h\|_{\Sigma} + |u_h|_{\text{👻}} \lesssim h^m \|u\|_{H^m(\Omega)} + \|\Pi^Q f^{\mathcal{E}} - f_h\|_{\Omega^{\mathcal{T}}}.$$

Theorem

For Ω smooth enough to assume L^2-H^2 regularity, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h \left(\|u_h - u\|_{L^2(\Omega)} + \gamma_{\text{👻}}^{1/2} |u|_{\text{👻}} \right) + \|f - f_h\|_{-2}.$$

Error estimates

Norm on $H(\operatorname{div}, \Omega^{\mathcal{T}})$:

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For Ω smooth enough to assume L^2-H^2 regularity, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h \left(\|u_h - u\|_{L^2(\Omega)} + \gamma_{\text{👻}}^{1/2} |u|_{\text{👻}} \right) + h^{m+1} \|f\|_{H^m}.$$

Post-processing

Motivation

Apply post-processing to

- repair inconsistency on cut elements
- obtain higher order convergence

Idea

Make use of the relation $\nabla p = u$ and exploit the accuracy of u_h.

Two versions

→ element-wise post-processing

→ patchwise post-processing

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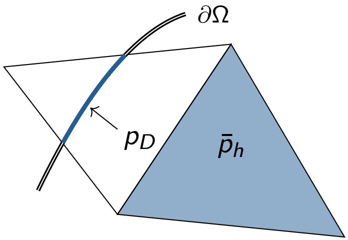
Element-local Scheme

For each $T \in \mathcal{T}_h$, find $p_h^* \in \mathbb{P}^{k+1}(T)$ s.t.

$$
\begin{aligned}
(\nabla p_h^*, \nabla q_h^*)_T &= (u_h, \nabla q_h^*)_T \quad && \forall q_h^* \in \mathbb{P}^{k+1}(T) \setminus \mathbb{R},\\
(p_h^*, 1)_T &= (\bar{p}_h, 1)_T && \text{if } T \in \mathcal{T}_h^{\text{interior}},\\
(p_h^*, 1)_{T\cap\partial\Omega} &= (p_D, 1)_{T\cap\partial\Omega} && \text{if } T \in \mathcal{T}_h^{\text{cut}}.
\end{aligned}
$$



Error estimate

If $\gamma_{👻} > 0$ and $p^{\mathcal{E}} \in H^{k+2}(\Omega^{\mathcal{T}})$, then

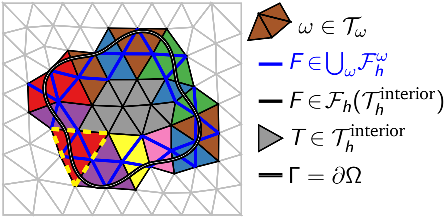
$$\|p^{\mathcal{E}} - p_h^*\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h^{k+2} \|p^{\mathcal{E}}\|_{H^{k+2}(\Omega^{\mathcal{T}})}.$$

Patchwise Post-processing

Patchwise Scheme

For each $\omega \in \mathcal{T}_\omega$, find $p_h^* \in \mathbb{P}^{k+1}(\omega)$ s.t. for all $q_h^* \in \mathbb{P}^{k+1}(\omega) \setminus \mathbb{R}$:

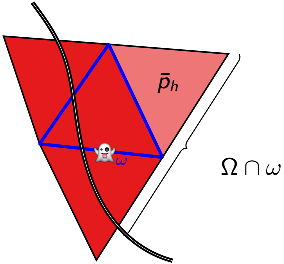
$$(\nabla p_h^*, \nabla q_h^*)_{\Omega \cap \omega} + \text{👻}_\omega(p_h^*, q_h^*) = (u_h, \nabla q_h^*)_{\Omega \cap \omega},$$
$$(p_h^*, 1)_{\Omega^{\text{interior}} \cap \omega} = (\bar{p}_h, 1)_{\Omega^{\text{interior}} \cap \omega}.$$



Error estimate

For $p \in H^{k+2}(\Omega)$, it holds

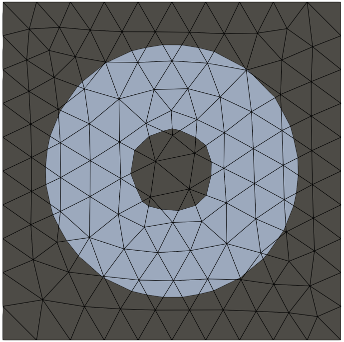
$$\|p - p_h^*\|_{L^2(\Omega)} \lesssim h^{k+2} \|p\|_{H^{k+2}(\Omega)}$$



- ➔ no dependence on Dirichlet boundary data
- ➔ $\gamma_{\text{👻}} = 0$ allowed (hybridization possible)

Numerical examples

- manufactured solution
- geometry description via levelset
- $\mathbb{RT}^k \times \mathbb{P}^k$
- isoparametric[3] unfitted FEM
- uniform refinements



[3] C. Lehrenfeld. High order unfitted finite element methods on level set domains using isoparametric mappings. CMAME 300, 2016.

Numerical examples: Approx. of u_h

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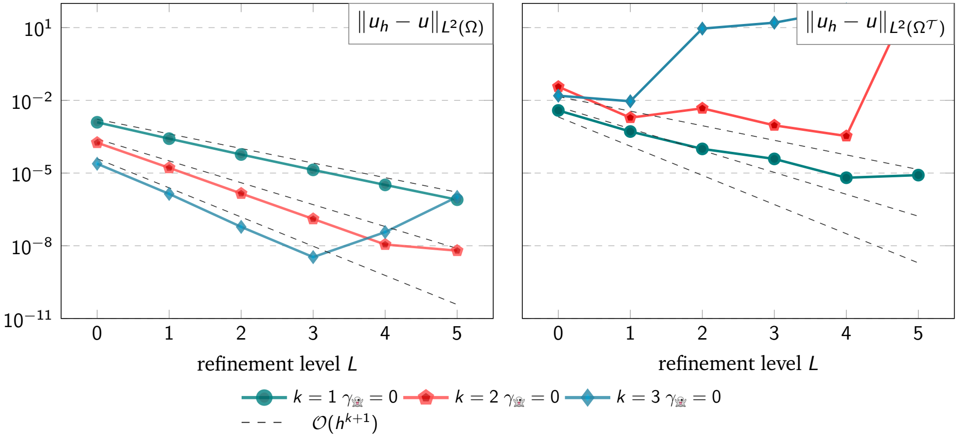
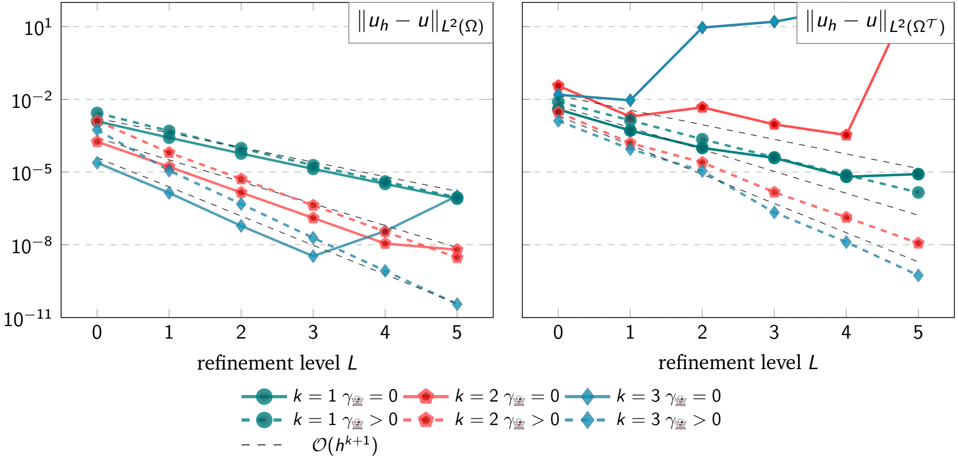
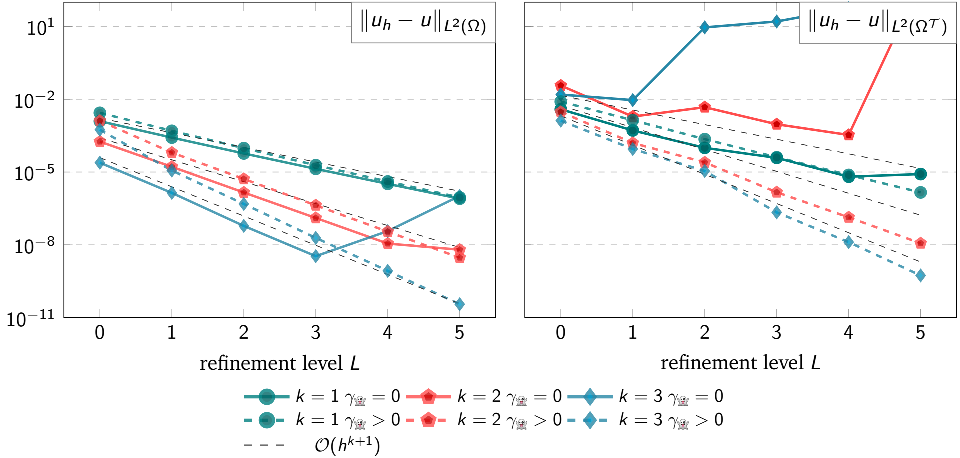


Numerical examples: Approx. of u_h

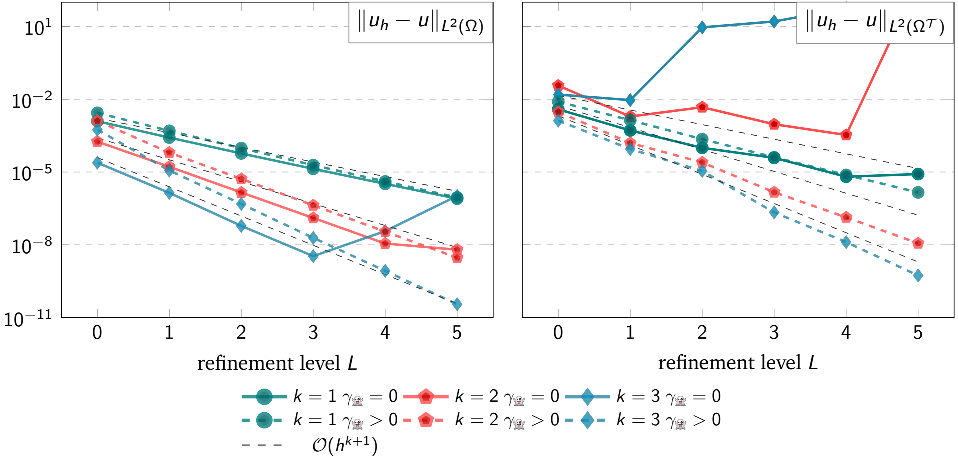


Numerical examples: Approx. of u_h



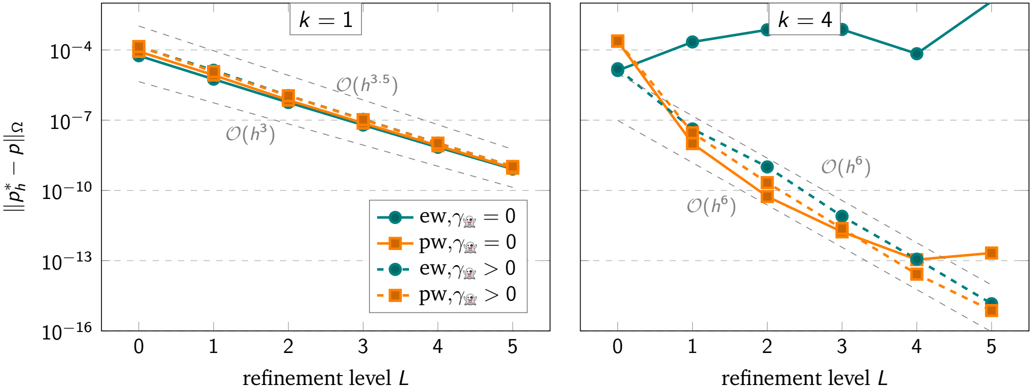
➔ for accuracy on $\Omega^{\mathcal{T}}$, $\gamma_{👻} > 0$ is necessary

Numerical examples: Approx. of u_h



- ➔ for accuracy on $\Omega^{\mathcal{T}}$, $\gamma_{👻} > 0$ is necessary
- ➔ for accuracy on Ω, $\gamma_{👻} = 0$ is possible but conditioning issues possible

Numerical examples: Post-processing



Conclusion

Unfitted Mixed FEM

- by extending b to b_h (Ω to $\Omega^{\mathcal{T}}$), we achieve inf-sup stability without polluting the mass balance
- post-processing allows to repair the inconsistency on cut elements and to obtain higher order convergence:
  - → Element-wise post processing: requires Dirichlet boundary data, $\gamma_{\text{👻}} > 0$ necessary
  - → Patchwise post processing: no dependence on boundary data, $\gamma_{\text{👻}} = 0$ allowed
- Hybridization possible ($\gamma_{\text{👻}} = 0$), but condition number unbounded

Got interested?

C. Lehrenfeld, TvB, I. Voulis, "Analysis of divergence-preserving unfitted finite element methods for the mixed Poisson problem" (2023), https://arxiv.org/pdf/2306.12722, (accepted for publication in Math. Comp.).